

Topology and Geometry of the Berkovich Ramification Locus for Rational Functions

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Contents

1	Introduction	2
2	Notation and Conventions	5
2.1	Non-Archimedean Fields	5
2.2	The Berkovich Projective Line	5
2.2.1	The Affine Line	5
2.2.2	The Weak Topology	6
2.2.3	The Strong Topology	7
2.2.4	The Projective Line	7
2.2.5	Tangent vectors	8
2.3	Rational Functions	8
2.3.1	Generalities	8
2.3.2	Rational functions over non-Archimedean fields	11
3	Multiplicity Functions	12
3.1	Extending m_φ to \mathbf{P}^1	12
3.2	The Directional Multiplicity	13
3.3	The Surplus Multiplicity	15
4	Extension of Scalars	18
5	The Locus of Inseparable Reduction	21
6	Connected Components	23
7	Endpoints and Interior Points	27
8	The Locus of Total Ramification	32

Abstract

Given a nonconstant holomorphic map $f : X \rightarrow Y$ between compact Riemann surfaces, one of the first objects we learn to construct is its ramification divisor R_f , which describes the locus at which f fails to be locally injective. The divisor R_f is a finite formal linear combination of points of X that is combinatorially constrained by the Hurwitz formula. Now let k be an algebraically closed field that is complete with respect to a nontrivial non-Archimedean absolute value. For example, $k = \mathbb{C}_p$. Here the role of a Riemann surface is played by a projective Berkovich analytic curve. As these curves have many points that are not algebraic over k , some new (non-algebraic) ramification behavior appears for maps between them. For example, the ramification locus is no longer a divisor, but rather a closed analytic subspace. This article initiates a detailed study of the ramification locus for self-maps $f : \mathbf{P}^1 \rightarrow \mathbf{P}^1$. This simplest first case has the benefit of being approachable by concrete (and often combinatorial) techniques.

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1 Introduction

Given a nonconstant holomorphic map $f : X \rightarrow Y$ between compact Riemann surfaces, one of the first objects we learn to construct is its ramification divisor R_f , which describes the locus at which f fails to be locally injective. The divisor R_f is a (finite) formal linear combination of points of X that is combinatorially constrained by the Hurwitz Formula: $2g_X - 2 = \deg(f)(2g_Y - 2) + \deg(R_f)$.

The goal of the present article is to initiate a study of the ramification locus in the setting of non-Archimedean analytic geometry. Here the role of a Riemann surface is played by a projective Berkovich analytic curve over a non-Archimedean field k . As these curves have many points that are not algebraic over k , some new (non-algebraic) ramification behavior appears. For example, the ramification locus is no longer a divisor, but rather a closed analytic subspace. Berkovich first observed this “geometric ramification” in [4, §6.3].

We begin our study by restricting attention to rational functions, viewed as endomorphisms of the projective line \mathbf{P}^1 . This simplest first case has the benefit of being approachable by concrete (and often combinatorial) techniques, many of which were developed by Rivera-Letelier [9, 10, 11], Favre/Rivera-Letelier [7], and Baker/Rumely [2]. As critical points occupy a central position in the study of complex dynamical systems on the Riemann sphere, it is not unreasonable to suppose that a better understanding of the Berkovich ramification locus for rational functions will have applications to non-Archimedean dynamical systems. In fact, this work was initially inspired by dynamical considerations in [7]. The simple structure of the ramification locus for tame polynomials plays a fundamental role in the recent work of Trucco [13]. We also expect that the nature of the ramification locus for dynamical systems defined over the formal Laurent series field $\mathbb{C}((t))$ will shed some light on degenerations of complex dynamical systems. See, e.g., [8].

Let k be an algebraically closed field that is complete with respect to a fixed nontrivial non-Archimedean absolute value $|\cdot|$. For example, k could be the completion of an algebraic closure of \mathbb{Q}_p or of $\mathbb{F}_p((t))$. The set of k -rational points of the projective line $\mathbb{P}^1(k) = k \cup \{\infty\}$, endowed with the metric topology from k , is neither connected nor locally compact. The Berkovich analytification $\mathbf{P}^1 = \mathbf{P}_k^1$ of the projective line was introduced to remedy these defects. It is a compact topological tree that contains a homeomorphic copy of $\mathbb{P}^1(k)$, and one may view it as a compactified parameter space of disks in $\mathbb{P}^1(k)$.

A rational function $\varphi \in k(z)$, viewed as a morphism $\varphi : \mathbb{P}_k^1 \rightarrow \mathbb{P}_k^1$, extends functorially to a morphism of \mathbf{P}^1 (which we also call φ). Intuitively, it describes the action of φ on disks in $\mathbb{P}^1(k)$.

As φ is a finite morphism, one may associate to each point $x \in \mathbf{P}^1$ a local degree or multiplicity $m_\varphi(x)$: in a (weak) neighborhood of x , the map φ is $m_\varphi(x)$ -to-1. The **(Berkovich) ramification locus** is defined to be the set

$$\mathcal{R}_\varphi = \{x \in \mathbf{P}^1 : m_\varphi(x) > 1\}.$$

The ramification locus is a closed subset of \mathbf{P}^1 with no isolated point. Our first main result provides a bound for the number of connected components \mathcal{R}_φ . We view this as a first step toward combinatorial constraints for the ramification locus.

Theorem A (Connected Components). Let $\varphi \in k(z)$ be a nonconstant rational function. Each connected component of the Berkovich ramification locus contains at least two critical points of φ (counted with weights).¹ In particular, \mathcal{R}_φ has at most $\deg(\varphi) - 1$ connected components.

The theorem is optimal in the following sense. For any algebraically closed field k that is complete with respect to a nontrivial non-Archimedean absolute value (as above) and any integers $1 \leq n < d$, there exists a rational function $\varphi \in k(z)$ of degree d whose ramification locus has precisely n connected components.

A field k as above always admits nontrivial extensions by non-Archimedean valued fields; it is one feature of non-Archimedean analysis that sets it apart from complex analysis. This observation is also particularly useful for simplifying various arguments in the subject. Let K/k be an extension of algebraically closed and complete non-Archimedean fields, which means the absolute value on K is an extension of the one on k . There is a natural inclusion $\mathbb{P}^1(k) \hookrightarrow \mathbb{P}^1(K)$, and this inclusion extends to the Berkovich analytifications $\iota : \mathbf{P}_k^1 \hookrightarrow \mathbf{P}_K^1$. However, this last map is not a morphism of analytic spaces (unless $K = k!$), and so we must spend some time defining it and proving that it preserves many of the features relevant to our study of ramification. In particular, we will show that this inclusion is continuous, that it preserves multiplicities, and that it preserves a certain natural metric on $\mathbf{P}^1 \setminus \mathbb{P}^1(k)$. The existence of the inclusion ι is closely related to Berkovich's notion of "peaked point;" see [3, 5.2].

A rational function $\varphi \in k(z)$ can act via an inseparable morphism on the local rings of certain points in \mathbf{P}^1 ; Rivera-Letelier calls this "inseparable reduction at a type II point." (The adjective "type II" refers to Berkovich's classification of points of \mathbf{P}^1 .) We give a natural extension of Rivera-Letelier's definition to all points of \mathbf{P}^1 . The idea is to enlarge the field k in such a way that all points of interest become type II points. This notion is independent of the choice of field extension.

The space \mathbf{P}^1 carries a second topology, called the strong topology. As an application of our work on extension of scalars and inseparable reduction, we give the following characterization of the strong interior of the ramification locus:

Theorem B (Interior Points). Let $\varphi \in k(z)$ be a nonconstant rational function.

1. The set of points at which φ has inseparable reduction coincides with the strong interior of the Berkovich ramification locus.
2. The ramification locus has empty weak interior unless φ is itself inseparable, in which case $\mathcal{R}_\varphi = \mathbf{P}^1$

¹For a rational function $\varphi \in k(z)$, a point at which the induced map on the tangent space of \mathbb{P}_k^1 vanishes will be called a **critical point**. The order of vanishing is called the **weight**.

We also look at the special setting of rational functions with a totally ramified point; i.e., a point $x \in \mathbf{P}^1$ such that $m_\varphi(x) = \deg(\varphi)$. For example, this includes the important cases of polynomials ($x = \infty$) and rational functions with good reduction (x is the Gauss point).

Theorem C (Totally Ramified Functions). Let $\varphi \in k(z)$ be a nonconstant rational function for which there exists a totally ramified point in \mathbf{P}^1 . Then the ramification locus \mathcal{R}_φ is connected. If $\text{res.char}(k) = 0$, or if $\text{res.char}(k) > \deg(\varphi)$, then $\mathcal{R}_\varphi = \text{Hull}(\text{Crit}(\varphi))$.

In a sequel to this paper, we provide a study of the geometry of the ramification locus [6]. The group $\text{PGL}_2(k)$ acts on \mathbf{P}^1 , and there is a natural “metric” ρ that is invariant under this action. (More precisely, ρ is a metric on the Berkovich hyperbolic space $\mathbf{P}^1 \setminus \mathbb{P}^1(k)$.) A rational function φ is locally expanding on its ramification locus for the metric ρ , and so it is desirable to control the size of \mathcal{R}_φ . For the following statement, let $\text{Hull}(\text{Crit}(\varphi))$ be the **connected hull** of the critical points; i.e., the smallest closed connected subset of \mathbf{P}^1 containing $\text{Crit}(\varphi)$. If $X \subset \mathbf{P}^1$ is a nonempty subset and $r \geq 0$ is a real number, define

$$X + r = \{y \in \mathbf{P}^1 : \rho(x, y) \leq r \text{ for some } x \in X\}.$$

Theorem D (Uniform Tubular Neighborhood). Suppose k has characteristic zero and residue characteristic $p \geq 0$. Let $\varphi \in k(z)$ be a nonconstant rational function. Then

$$\mathcal{R}_\varphi \subset \text{Hull}(\text{Crit}(\varphi)) + \begin{cases} 0 & \text{if } p = 0 \text{ or } p > \deg(\varphi) \\ \frac{1}{p-1} & \text{if } 0 < p \leq \deg(\varphi). \end{cases}$$

As an application, we deduce a non-Archimedean version of Rolle’s Theorem for rational functions. We also obtain an analogous result when k has positive characteristic and φ is tamely ramified, although the bound cannot be made uniformly as one varies the rational function φ .

We hope to extend this study in two directions. First, it is desirable to give a combinatorial characterization of the pairs (X, m) consisting of a closed subset $X \subset \mathbf{P}^1$ and a function $m : X \rightarrow \{1, \dots, d\}$ such that $(X, m) \cong (\mathcal{R}_\varphi, m_\varphi)$ for some rational function $\varphi \in k(z)$ of degree d . (Examples suggest that \cong should mean there is a ρ -isometric bijection $X \rightarrow \mathcal{R}_\varphi$ that is compatible with m and m_φ .) The present article contributes a large step in this direction, and we have made significant further progress when $p = 0$ or $p > d$ — i.e., the cases in which \mathcal{R}_φ is a finite graph. It is reasonable to expect that such an investigation will lead to some understanding of the geometry of the space of rational functions with a fixed ramification type (X, m) . Although much of the machinery we use in this article is special to the projective line, we also hope to find a suitable extension of the main ideas to the setting of finite morphisms $\mathbf{X} \rightarrow \mathbf{Y}$ of projective analytic curves.

We close with a detailed summary of the contents of the paper. In Section 2 we recall all of the relevant features of \mathbf{P}^1 and its endomorphisms. While this section is primarily designed to fix notation, it could also serve as a brief introduction to \mathbf{P}^1 . In Section 3 we discuss three notions of multiplicity function. The first is an extension of the algebraic multiplicity m_φ on $\mathbb{P}^1(k)$ to the entire Berkovich projective line \mathbf{P}^1 . The second is the directional multiplicity, which allows one to accurately count the number of solutions to the equation $\varphi(z) = y$ in a particular open Berkovich disk U , provided that $\varphi(U) \neq \mathbf{P}^1$. It can happen that $\varphi(U) = \mathbf{P}^1$, and so we introduce the notion of surplus multiplicity as the defect in this counting problem. The surplus multiplicity of U is very closely tied to the number of critical points contained in U . The first two multiplicities are well-understood. This article is the first to focus on the surplus multiplicity in its own right, although it does appear in [11, Lem. 3.2].

Section 4 is devoted to constructing the canonical inclusion $\iota_k^K : \mathbf{P}_k^1 \rightarrow \mathbf{P}_K^1$ and proving a number of useful properties, including its compatibility with rational functions. The goal of Section 5 is to provide a definition of inseparable reduction at an arbitrary point of \mathbf{P}^1 . We also give an interesting criterion for when a type III point has inseparable reduction. In Section 6, we prove Theorem A and a number of other results related to connectedness of the ramification locus. For example, we show that every connected component of \mathcal{R}_φ meets the convex hull of the critical points. We describe the endpoints and interior points of the ramification locus in Section 7; this includes a proof of Theorem B. Finally, in Section 8 we discuss the locus of total ramification and some of the properties of rational functions for which this locus is nonempty.

2 Notation and Conventions

2.1 Non-Archimedean Fields

For the duration of this paper, k will denote an algebraically closed field that is complete with respect to a nontrivial non-Archimedean absolute value $|\cdot|$. Examples to keep in mind are the completion of an algebraic closure of \mathbb{Q}_p (denoted \mathbb{C}_p) or of the formal Laurent series field $L((T))$, where L is a field and $|f|_L = \exp(-\text{ord}_{T=0}(f))$ for $f \in L((T))$.

We use the standard notation $k^\circ = \{t \in k : |t| \leq 1\}$ and $k^{\circ\circ} = \{t \in k : |t| < 1\}$ for the valuation ring of k and for its maximal ideal, respectively, and we write $\tilde{k} = k^\circ/k^{\circ\circ}$ for the residue field. The residue characteristic of k will be denoted p . (Note $p = 0$ is allowed.) The value group of k will be denoted $|k^\times|$; as k is algebraically closed, $|k^\times|$ is a divisible group.

The **normalized base** associated to k is the constant

$$q_k = \begin{cases} e & \text{if } k \text{ has equicharacteristic } p \geq 0 \\ |p|^{-1} & \text{if } k \text{ has mixed characteristic.} \end{cases}$$

Then $q_k > 1$, and the function $\text{ord}_k(\cdot) = -\log_{q_k} |\cdot|$ is a valuation on k . The choice of normalization is unimportant in the equicharacteristic case. When k has mixed characteristic, the constant $1/(p-1)$ appearing in Theorem ?? depends on our choice of normalized base. More precisely, note that $\text{ord}_k(p^{1/(p-1)}) = 1/(p-1)$.

For $a \in k$ and $r \in \mathbb{R}_{\geq 0}$, write

$$D(a, r)^- = \{x \in k : |x - a| < r\} \quad \text{and} \quad D(a, r) = \{x \in k : |x - a| \leq r\}$$

for the **(classical) open disk** and the **(classical) closed disk** of radius r about a , respectively.

2.2 The Berkovich Projective Line

Here we summarize the definition and main properties of \mathbf{P}^1 . For the most part we follow the notation and treatment in [2, §1–2], although much of this material was first presented in [10, 11]. See also [1, Ch. 3].

2.2.1 The Affine Line

The Berkovich affine line $\mathbf{A}^1 = \mathbf{A}_k^1$ is defined to be the set of all multiplicative seminorms on the polynomial algebra $k[T]$ that restrict to the given absolute value on k . If x is a seminorm and

$f \in k[T]$ is a polynomial, we write $|f(x)|$ for the value of f at x . For example, if $a \in k$ and $r \in \mathbb{R}_{\geq 0}$, write $\zeta_{a,r}$ for the multiplicative seminorm defined by

$$|f(\zeta_{a,r})| = \sup_{b \in D(a,r)} |f(b)|, \quad f \in k[T].$$

Berkovich has classified the points of \mathbf{A}^1 :

1. Type I. $\zeta_{a,0}$ for some $a \in k$. (Such a point is called a **classical point**.)
2. Type II. $\zeta_{a,r}$ for some $a \in k$ and $r \in |k^\times|$.
3. Type III. $\zeta_{a,r}$ for some $a \in k$ and $r \notin |k^\times|$.
4. Type IV. A limit of seminorms $(\zeta_{a_i, r_i})_{i \geq 0}$, where the associated sequence of closed disks $(D(a_i, r_i))_{i \geq 0}$ is descending and has empty intersection. (The field k is called **spherically closed** if no such sequence of closed disks exists.)

This classification suggests a means for extending the notation to cover type IV points. Given a decreasing sequence of closed disks $D(\mathbf{a}, \mathbf{r}) = (D(a_i, r_i))_{i \geq 0}$, define $\zeta_{\mathbf{a}, \mathbf{r}} \in \mathbf{A}^1$ to be the seminorm on $k[T]$ given by

$$|f(\zeta_{\mathbf{a}, \mathbf{r}})| = \lim_{i \rightarrow \infty} \sup_{b \in D(a_i, r_i)} |f(b)|, \quad f \in k[T].$$

Note that $\zeta_{\mathbf{a}, \mathbf{r}} = \zeta_{a,r}$ if $D(\mathbf{a}, \mathbf{r})$ is the constant sequence with term $D(a, r)$. More generally, if $\cap_{i \geq 0} D(a_i, r_i) = D(b, s)$ for some $b \in k$ and $s \in \mathbb{R}_{\geq 0}$, then one verifies easily that $\zeta_{\mathbf{a}, \mathbf{r}} = \zeta_{b,s}$. Moreover, we have the equality of seminorms $\zeta_{\mathbf{a}, \mathbf{r}} = \zeta_{\mathbf{a}', \mathbf{r}'}$ if and only if the associated sequences $D(\mathbf{a}, \mathbf{r})$ and $D(\mathbf{a}', \mathbf{r}')$ are cofinal in each other.

We identify the set of classical points in \mathbf{A}^1 with k via the injection $a \mapsto \zeta_{a,0}$. The point $\zeta_{0,1}$ is called the **Gauss point** because the associated seminorm coincides with the Gauss norm of a polynomial.

2.2.2 The Weak Topology

The **weak topology** on \mathbf{A}^1 is the weakest topology satisfying the following property: for each polynomial $f \in k[T]$, the function $x \mapsto |f(x)|$ is continuous on \mathbf{A}^1 . The space \mathbf{A}^1 is locally compact, Hausdorff, and uniquely path-connected for the weak topology.

The injection $k \hookrightarrow \mathbf{A}^1$ given by $a \mapsto \zeta_{a,0}$ is a dense homeomorphic embedding relative to the absolute value topology on k and the weak topology on \mathbf{A}^1 . The type II points of \mathbf{A}^1 are dense in \mathbf{A}^1 for the weak topology.

For $a \in k$ and $r \in \mathbb{R}_{\geq 0}$, the sets

$$\mathcal{D}(a, r)^- = \{x \in \mathbf{A}^1 : |(T - a)(x)| < r\} \quad \text{and} \quad \mathcal{D}(a, r) = \{x \in \mathbf{A}^1 : |(T - a)(x)| \leq r\}$$

are the **(standard) open Berkovich disk** and the **(standard) closed Berkovich disk** of radius r about a , respectively. It is easy to see that the classical disks in k may be recovered via $D(a, r)^- = \mathcal{D}(a, r)^- \cap k$ and similarly for $D(a, r)$. The weak topology on \mathbf{A}^1 is generated by sets of the form

$$\mathcal{D}(a, r)^- \text{ and } \mathbf{A}^1 \setminus \mathcal{D}(a, r)$$

for $a \in k$ and $r \in \mathbb{R}_{>0}$.

2.2.3 The Strong Topology

The space \mathbf{A}^1 carries another topology, called the **strong topology**, that is strictly finer than the weak topology. As we do not need its definition in the present article, we are content to point the reader to [2, §2.7] as a reference. Instead, we will make use of a metric ρ on the **Berkovich hyperbolic space** $\mathbf{H} = \mathbf{A}^1 \setminus k$ that induces the restriction of the strong topology.

The affine line \mathbf{A}^1 admits a partial ordering \preceq defined by $x \preceq y$ if and only if $|f(x)| \leq |f(y)|$ for all polynomials $f \in k[T]$. For example, $\zeta_{a,r} \preceq \zeta_{b,s}$ if and only if $D(a,r) \subset D(b,s)$. Given $x, y \in \mathbf{A}^1$, the least upper bound with respect to the partial ordering is denoted $x \vee y$. (It always exists and is unique.) Type I and type IV points are the minimal elements with respect to \preceq .

Define the **affine diameter** of the point $\zeta_{a,r}$ to be $\text{diam}(\zeta_{a,r}) = r$. More generally, if (ζ_{a_i, r_i}) is a sequence of seminorms corresponding to a type IV point $x \in \mathbf{A}^1$, define $\text{diam}(x) = \lim r_i$. The limit exists since (r_i) is a decreasing sequence, and $\text{diam}(x) > 0$ (else this sequence corresponds to a type I point).

Now we define the **path-distance metric** ρ on $\mathbf{H} = \mathbf{A}^1 \setminus k$ via the formula

$$\begin{aligned} \rho(x, y) &= 2 \log_{q_k} \text{diam}(x \vee y) - \log_{q_k} \text{diam}(x) - \log_{q_k} \text{diam}(y) \\ &= \log_{q_k} \frac{\text{diam}(x \vee y)}{\text{diam}(x)} + \log_{q_k} \frac{\text{diam}(x \vee y)}{\text{diam}(y)}. \end{aligned}$$

The restriction of the strong topology to \mathbf{H} coincides with the metric topology for ρ . The space \mathbf{H} is complete for this metric, but not locally compact. Note that our choice of normalized base q_k gives $\rho(\zeta_{0,q_k}, \zeta_{0,1}) = 1$. We extend ρ to the entirety of \mathbf{P}^1 by defining $\rho(x, y) = +\infty$ whenever x or y is a type I point. Note that this extension does not give the strong topology on \mathbf{P}^1 .

The group $\text{PGL}_2(k)$ acts by isometries for the path-distance metric: $\rho(\sigma(x), \sigma(y)) = \rho(x, y)$ for any $x, y \in \mathbf{H}$ and $\sigma \in \text{PGL}_2(k)$.

2.2.4 The Projective Line

The Berkovich projective line over k , denoted $\mathbf{P}^1 = \mathbf{P}_k^1$, is given by gluing two copies of \mathbf{A}^1 along $\mathbf{A}^1 \setminus \{0\}$ via the map $T \mapsto 1/T$. The weak topology on \mathbf{P}^1 is induced by this gluing. We write $\{\infty\} = \mathbf{P}^1 \setminus \mathbf{A}^1$, and the dense homeomorphic embedding $k \hookrightarrow \mathbf{A}^1$ extends to $\mathbb{P}^1(k) \hookrightarrow \mathbf{P}^1$. We also extend the partial ordering \preceq to \mathbf{P}^1 by setting $x \preceq \infty$ for every $x \in \mathbf{P}^1$. For $x \preceq x' \in \mathbf{P}^1$, we define the **closed segment** $[x, x'] = \{y \in \mathbf{P}^1 : x \preceq y \preceq x'\}$, and extend this notion to arbitrary pairs $x, x' \in \mathbf{P}^1$ by $[x, x'] = [x, x \vee x'] \cup [x', x \vee x']$. Open and half-open segments can be defined similarly.

The group $\text{PGL}_2(k)$ acts on $\mathbb{P}^1(k)$, and this action extends functorially to \mathbf{P}^1 . Moreover, the action preserves the type of a point in \mathbf{P}^1 , and it is transitive on the set of type I and type II points. The image of an open disk $\mathcal{D}(a, r)^- \subset \mathbf{A}^1$ under the action of an element of $\text{PGL}_2(k)$ will be called an **open Berkovich disk** (and similarly for a **closed Berkovich disk**). The weak topology on \mathbf{P}^1 is generated by sets of the form $\mathcal{D}(a, r)^-$ and $\mathbf{P}^1 \setminus \mathcal{D}(a, r)$ for $a \in k$ and $r \in \mathbb{R}_{>0}$. The space \mathbf{P}^1 is compact, Hausdorff, and uniquely path-connected for the weak topology.

We close with the following important property of the strong and weak topologies on \mathbf{P}^1 :

Proposition 2.1 ([2, Lem. B.18]). *Let $X \subset \mathbf{P}^1$ be a subset. Then X is connected for the weak topology on \mathbf{P}^1 if and only if it is connected for the strong topology on \mathbf{P}^1 .*

Consequently, we may speak of the connected components of a subset $X \subset \mathbf{P}^1$ without reference to the topology.

2.2.5 Tangent vectors

Let $x \in \mathbf{P}^1$ be a point. Write T_x for the set of connected components of $\mathbf{P}^1 \setminus \{x\}$; an element $\vec{v} \in T_x$ will be called a **tangent vector** at x . If we wish to view a connected component $\vec{v} \in T_x$ as a subset of \mathbf{P}^1 , then we will write it as $\mathcal{B}_x(\vec{v})^-$. Observe that the weak topology on \mathbf{P}^1 is generated by the sets $\mathcal{B}_x(\vec{v})^-$ as x varies through \mathbf{P}^1 and \vec{v} varies through T_x .

The cardinality of the set T_x depends only on the type of the point x :

1. Type I. T_x consists of a single tangent vector.
2. Type II. T_x is in 1-to-1 correspondence with elements of $\mathbb{P}^1(\tilde{k})$.
3. Type III. T_x consists of two tangent vectors.
4. Type IV. T_x consists of a single tangent vector.

In the case of a type II point x , the correspondence between T_x and $\mathbb{P}^1(\tilde{k})$ is non-canonical except when $x = \zeta_{0,1}$. The correspondence $\mathbb{P}^1(\tilde{k}) \xrightarrow{\sim} T_{\zeta_{0,1}}$ is given by $a \mapsto \vec{a}$, where \vec{a} is the connected component of $\mathbf{P}^1 \setminus \{\zeta_{0,1}\}$ all of whose classical points map to a under the canonical reduction map $\mathbb{P}^1(k) \rightarrow \mathbb{P}^1(\tilde{k})$.

2.3 Rational Functions

2.3.1 Generalities

Let L be an algebraically closed field, and let $\varphi \in L(z)$ be a nonconstant rational function. Choose polynomials $f, g \in L[z]$ with no common root such that $\varphi = f/g$. Write $\deg(\varphi) = \max\{\deg(f), \deg(g)\}$.

Suppose $x \in \mathbb{P}^1(L)$ and set $y = \varphi(x)$. Select $\sigma_1, \sigma_2 \in \text{PGL}_2(L)$ such that $\sigma_1(0) = x$ and $\sigma_2(y) = 0$, and define $\psi = \sigma_2 \circ \varphi \circ \sigma_1$. The **multiplicity** of φ at x is defined to be the integer $m_\varphi(x) = \text{ord}_{z=0} \psi(z)$. Evidently $1 \leq m_\varphi(x) \leq \deg(\varphi)$. The **weight** of φ at x is defined as $w_\varphi(x) = \text{ord}_{z=0} \psi'(z)$. If $\varphi'(z) \equiv 0$, we set $w_\varphi(x) = +\infty$. The weight and multiplicity at x are independent of the choice of σ_1 and σ_2 .

If $\varphi(x) = y$ with $x, y \neq \infty$, then one verifies that

$$m_\varphi(x) = \text{ord}_{z=x} (\varphi(z) - y) \quad w_\varphi(x) = \text{ord}_{z=x} (\varphi'(z)).$$

As an immediate consequence, we obtain the following formula for each $y \in \mathbb{P}^1(L)$:

$$\sum_{\substack{x \in \mathbb{P}^1(L) \\ \varphi(x)=y}} m_\varphi(x) = \deg(\varphi).$$

Remark 2.2. In some of the literature, the multiplicity $m_\varphi(x)$ is referred to as the “ramification index” or as the “local degree.” The weight $w_\varphi(x)$ is a non-standard terminology special to this paper; it is referred to as the “multiplicity” of a critical point in most of the literature. As our focus is on certain multiplicity functions, we have chosen an alternative terminology to avoid confusion.

Let p be the characteristic of L . The weight and multiplicity of a point are related by

$$w_\varphi(x) \begin{cases} = m_\varphi(x) - 1 & \text{if } p \nmid m_\varphi(x) \\ > m_\varphi(x) - 1 & \text{if } p \mid m_\varphi(x). \end{cases}$$

We say that φ is **ramified** (resp. **unramified**) at x if $m_\varphi(x) > 1$ (resp. $m_\varphi(x) = 1$). If $p \mid m_\varphi(x)$, we say that φ is **wildly ramified** at x ; otherwise φ is **tamely ramified** at x . A point x with positive weight is called a **critical point** of φ ; the above relations between weights and multiplicities show that φ is ramified at x if and only if x is a critical point. We write $\text{Crit}(\varphi)$ for the set of critical points of φ .

If L has characteristic $p > 0$, a rational function $\varphi \in L(z)$ is called **inseparable** if $\varphi(z) = \psi(z^p)$ for some rational function ψ . Otherwise φ is said to be **separable**. (Equivalently, φ is separable if and only if the extension of fields $L(z)/L(\varphi(z))$ is separable.) We will need a number of alternative descriptions of inseparability. (See also [11, §4.1].)

Proposition 2.3. *Suppose L has characteristic $p > 0$, and let $\varphi \in L(z)$ be a nonconstant rational function. The following statements are equivalent:*

1. φ is inseparable.
2. $\sigma_2 \circ \varphi \circ \sigma_1$ is inseparable for any $\sigma_1, \sigma_2 \in \text{PGL}_2(L)$.
3. $\varphi' = 0$.
4. φ has infinitely many critical points.
5. Each point of $\mathbb{P}^1(L)$ is a critical point for φ .

Proof. The equivalence of (1) and (2) follows from the definition and the fact that $\left(\frac{az+b}{cz+d}\right)^p = \frac{a^p z^p + b^p}{c^p z^p + d^p}$ for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PGL}_2(L)$. Note that the numerator of the rational function φ' is a polynomial, and hence either has finitely many zeros or else vanishes identically. Since a point $x \neq \infty$ is critical if and only if the derivative of φ vanishes at x , we see that (3) is equivalent to each of (4) and (5).

It remains to show that (1) and (3) are equivalent. If φ is inseparable, then $\varphi(z) = \psi(z^p)$ for some rational function ψ . Hence $\varphi'(z) = pz^{p-1}\psi'(z^p) = 0$. Conversely, suppose that $\varphi' = 0$. Writing $\varphi = f/g$ for some polynomials f, g of minimal degree, the quotient rule for derivatives shows $fg' = f'g$. Now $g' = 0$, for otherwise we may write $f/g = f'/g'$ in contradiction to the fact that f, g were chosen to have minimal degree. It follows that $g(z) = g_1(z^p)$ for some polynomial $g_1 \in L[z]$. Similarly, $f' = 0$, and we may write $f(z) = f_1(z^p)$. Set $\psi = f_1/g_1$ to see that $\varphi(z) = \psi(z^p)$. \square

Proposition 2.4. (*Hurwitz Formula*) *Let $\varphi \in L(z)$ be a nonconstant rational function. The collection of weights for φ are related by*

$$\sum_{x \in \mathbb{P}^1(L)} w_\varphi(x) = \begin{cases} 2 \deg(\varphi) - 2 & \text{if } \varphi \text{ is separable} \\ +\infty & \text{if } \varphi \text{ is inseparable.} \end{cases}$$

In particular, a nonconstant separable rational function has at most $2 \deg(\varphi) - 2$ distinct critical points.

For a nonconstant rational function $\varphi \in L(z)$, choose polynomials f, g with no common root such that $\varphi = f/g$. This choice is unique up to a common nonzero factor in L . The **Wronskian** of $\varphi = f/g \in L(z)$ is defined to be

$$\text{Wr}_\varphi = f'g - fg' \in L[z].$$

It is a polynomial of degree at most $2 \deg(\varphi) - 2$ whose roots are precisely the affine critical points of φ . (If one wants to recover all critical points, then one should work with the homogeneous Wronskian $Y^{2 \deg(\varphi) - 2} \text{Wr}_\varphi(X/Y) \in L[X, Y]$.) The Wronskian depends on the choice of representation f/g , although we suppress this from the notation.

Proof of Proposition 2.4. We have already observed above that $w_\varphi(x) \geq p > 0$ for every x when φ is inseparable. An algebraically closed field has infinite cardinality, so the formula follows in this case.

Now suppose that φ is separable, so that φ has only finitely many critical points. Without loss of generality we may change coordinates on the source and target so that ∞ is a non-critical fixed point of φ . Choose polynomials $f, g \in L[z]$ with no common root such that $\varphi = f/g$, and write $f(z) = a_d z^d + \cdots + a_0$ and $g(z) = b_d z^d + \cdots + b_0$ with $a_i, b_j \in L$. Then $b_d = 0$ and $a_d \neq 0$. The fact that ∞ is not a critical point is equivalent to saying that $b_{d-1} \neq 0$; this may be observed, for example, by computing the derivative of $1/\varphi(1/z)$. A direct calculation shows the Wronskian $\text{Wr}_\varphi(z)$ takes the form

$$\text{Wr}_\varphi(z) = a_d b_{d-1} z^{2d-2} + \cdots.$$

Observe that for $x \in L$,

$$\text{ord}_{z=x} \text{Wr}_\varphi(z) = \text{ord}_{z=x} \varphi'(z) = w_\varphi(x).$$

Hence

$$\sum_{x \in \mathbb{P}^1(L)} w_\varphi(x) = \sum_{x \in L} \text{ord}_{z=x} \text{Wr}_\varphi(z) = \deg \text{Wr}_\varphi = 2 \deg(\varphi) - 2.$$

□

Corollary 2.5. *Let φ be a separable rational function of degree at least 2. Either there exist at least two distinct critical points for φ , or else L has positive characteristic p and φ is wildly ramified at a single point c with weight $2 \deg(\varphi) - 2$.*

Proof. Suppose that c is the only critical point for φ . If L has characteristic zero, or if L has characteristic $p > 0$ and $p \nmid m_\varphi(c)$, then the Hurwitz formula shows that

$$2 \deg(\varphi) - 2 = w_\varphi(c) = m_\varphi(c) - 1 \leq \deg(\varphi) - 1.$$

But this is absurd since $\deg(\varphi) > 1$. Hence L must have characteristic $p > 0$ and φ must be wildly ramified at c . □

To close this section, we derive an explicit formula for the Wronskian Wr_φ in terms of the coefficients of f and g . Let $d = \deg(\varphi)$ and write

$$f(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0, \quad g(z) = b_d z^d + b_{d-1} z^{d-1} + \cdots + b_0,$$

for some coefficients $a_i, b_j \in L$. Let us make the convention that $a_i = b_i = 0$ if $i < 0$ or $i > d$. Then the Wronskian of $\varphi = f/g$ is given by

$$\text{Wr}_\varphi(z) = f'(z)g(z) - f(z)g'(z) = \sum_{i \geq 0} \sum_{j \geq 0} (ia_i b_j - ja_i b_j) z^{i+j-1}.$$

Making the change of variable $j \mapsto j - i + 1$ gives

$$\text{Wr}_\varphi(z) = \sum_{j \geq 0} \left\{ \sum_{i \geq 0} (2i - j - 1) a_i b_{j+1-i} \right\} z^j. \quad (2.1)$$

2.3.2 Rational functions over non-Archimedean fields

A rational function $\varphi \in k(z)$, viewed as an endomorphism of $\mathbb{P}^1(k)$, extends functorially to an endomorphism of \mathbf{P}^1 . By abuse of notation, we denote the extension by φ as well. The map $\varphi : \mathbf{P}^1 \rightarrow \mathbf{P}^1$ is continuous for both the weak and strong topologies.

Intuitively, the extension $\varphi : \mathbf{P}^1 \rightarrow \mathbf{P}^1$ reflects the mapping properties of open disks in $\mathbb{P}^1(k)$. More precisely, if φ is nonconstant, we can describe the extension of φ to type II points in the following concrete fashion. Let $S \subset k^\circ$ be a complete collection of coset representatives for $\tilde{k} = k^\circ/k^{\circ\circ}$. The closed disk $D(0, 1)$ is a disjoint union of open disks $D(b, 1)^-$ as b varies through S . For all but finitely many $b \in S$, the image $\varphi(D(b, 1)^-)$ is an open disk $D(\varphi(b), s)^-$ for some $s \in |k^\times|$. For any such choice of b , we have $\varphi(\zeta_{0,1}) = \zeta_{\varphi(b), s}$. For an arbitrary type II point $\zeta_{a,r}$, choose $\sigma \in \text{PGL}_2(k)$ so that $\sigma(\zeta_{0,1}) = \zeta_{a,r}$, and apply the preceding discussion to the rational function $\varphi \circ \sigma$.

Let $\varphi \in k(z)$ be a nonconstant rational function. We may write $\varphi = f/g$ for polynomials $f, g \in k[z]$ with no common root. If $f, g \in k^\circ[z]$ and if the maximum absolute value of the coefficients of f and g is 1, then we say φ is **normalized**.

Given $\varphi \in k(z)$, we may always choose polynomials $f, g \in k^\circ[z]$ so that $\varphi = f/g$ is normalized. (It can be accomplished by dividing the numerator and denominator of an arbitrary representation by a judicious choice of nonzero element of k .) This choice of f and g is unique up to simultaneous multiplication by an element in k with absolute value 1. Write \tilde{f} and \tilde{g} for the images of f and g in $k^\circ[z]/k^{\circ\circ}[z]$, respectively. The **reduction** of φ is given by

$$\tilde{\varphi}(z) = \begin{cases} \tilde{f}/\tilde{g} & \text{if } g \notin k^{\circ\circ}[z] \\ \infty & \text{if } g \in k^{\circ\circ}[z]. \end{cases}$$

The degree of $\tilde{\varphi}$ is independent of the choice of normalized representation $\varphi = f/g$. (By convention, we set $\deg(\infty) = 0$.) We say that φ has **constant reduction** (resp. **nonconstant reduction**) if the degree of $\tilde{\varphi}$ is zero (resp. positive).

Proposition 2.6. *Let $\varphi \in k(z)$ be a nonconstant rational function, and write $\varphi = f/g$ in normalized form. Then φ has nonconstant reduction if and only if $\varphi(\zeta_{0,1}) = \zeta_{0,1}$.*

Proof. This is essentially Lemma 2.17 of [2]. As the point at infinity plays a distinguished role in much of their theory, they do not treat the case in which φ has constant reduction with value ∞ . This issue can be remedied by replacing $\varphi(z)$ with $1/\varphi(1/z)$. \square

Let x be a point of \mathbf{P}^1 and \vec{v} a tangent direction at x . Then for every $y \in \mathcal{B}_x(\vec{v})^-$ sufficiently close to x , the image segment $\varphi((x, y))$ does not contain $\varphi(x)$, and hence it lies entirely in a single connected component of $\mathbf{P}^1 \setminus \{\varphi(x)\}$. In this way, φ determines a surjective map $\varphi_* : T_x \rightarrow T_{\varphi(x)}$ [2, Cor. 9.20]. We have already seen that $T_{\zeta_{0,1}}$ is canonically identified with $\mathbb{P}^1(\tilde{k})$. If $\varphi(\zeta_{0,1}) = \zeta_{0,1}$, then under this identification we have $\tilde{\varphi} = \varphi_*$.

3 Multiplicity Functions

3.1 Extending m_φ to \mathbf{P}^1

Here we describe an extension of the multiplicity function m_φ on $\mathbb{P}^1(k)$ to the Berkovich projective line \mathbf{P}^1 , where k is a non-Archimedean field. There are a number of equivalent ways to do this; see [2, §9.1], [4, §6.3.1], and [7, §2.2]. The definition is relatively unimportant for our purposes in this paper (although we give one for completeness); instead, we rely on various characterizations and properties of the multiplicity function to be recalled below.

The most direct definition of the multiplicity function is as follows. Let k be a non-Archimedean field, let $\varphi \in k(z)$ be a nonconstant rational function, and let $\mathcal{O}_{\mathbf{P}^1}$ be the analytic structure sheaf on \mathbf{P}^1 . Then $\varphi_* \mathcal{O}_{\mathbf{P}^1}$ is a locally free $\mathcal{O}_{\mathbf{P}^1}$ module. The **multiplicity** of φ at $x \in \mathbf{P}^1$ is defined as

$$m_\varphi(x) = \mathrm{rk}_{\mathcal{O}_{\mathbf{P}^1, y}}(\varphi_* \mathcal{O}_{\mathbf{P}^1})_x = \mathrm{rk}_{\mathcal{O}_{\mathbf{P}^1, y}} \mathcal{O}_{\mathbf{P}^1, x},$$

where $y = \varphi(x)$.

More intuitively, we have the following topological characterization that appears in the work of Rivera-Letelier. It will be the first instance of many in which we want to count a set of points “with multiplicities.” To be precise, if $X \subset \mathbf{P}^1$ is a set, then to count X **with multiplicities** means to compute the quantity

$$\#X = \sum_{x \in X} m_\varphi(x).$$

Proposition 3.1 ([2, Cor. 9.17]). *For each $x \in \mathbf{P}^1$ and for each sufficiently small φ -saturated neighborhood U of x (i.e., U is a connected component of $\varphi^{-1}(\varphi(U))$), the multiplicity $m_\varphi(x)$ is equal to $\#U \cap \varphi^{-1}(\{b\})$ for each $b \in \varphi(U) \cap \mathbb{P}^1(k)$.*

Intuitively, this says that each classical point near $\varphi(x)$ has $m_\varphi(x)$ pre-images when counted with multiplicities. More generally, it is true that if $x \in \mathbf{P}^1$ has multiplicity $m = m_\varphi(x)$, then the map φ is locally m -to-1 in a neighborhood of x , provided that we count with multiplicities. The function $m_\varphi : \mathbf{P}^1 \rightarrow \{1, \dots, \deg(\varphi)\}$ is sometimes called the “local degree function” for this reason.

Definition 3.2. Let $\varphi \in k(z)$ be a nonconstant rational function. The **(Berkovich) ramification locus** for φ is the set

$$\mathcal{R}_\varphi = \{x \in \mathbf{P}^1 : m_\varphi(x) > 1\}.$$

Remark 3.3. We call an arbitrary point $x \in \mathcal{R}_\varphi$ a **ramified point**, while we reserve the term “critical point” for the type I points in \mathcal{R}_φ . (A different convention is used in [13].)

A rational function φ of degree 1 is an automorphism, and so is everywhere injective. Proposition 3.4(2) below shows m_φ is identically 1 on \mathbf{P}^1 , so that \mathcal{R}_φ is empty. A rational function φ with $\deg(\varphi) \geq 2$ has a critical point — i.e., a classical point of multiplicity at least 2 — and so \mathcal{R}_φ is nonempty.

Proposition 3.4 ([2, Prop. 9.28]). *Let $\varphi \in k(z)$ be a nonconstant rational function. The multiplicity function $m_\varphi : \mathbf{P}^1 \rightarrow \{1, \dots, \deg(\varphi)\}$ enjoys the following properties.*

1. *m_φ is upper semicontinuous with respect to the weak topology. That is, the set $\{x \in \mathbf{P}^1 : m_\varphi(x) \geq i\}$ is weakly closed in \mathbf{P}^1 for each $i = 1, 2, \dots, \deg(\varphi)$.*
2. *The map $\varphi : \mathbf{P}^1 \rightarrow \mathbf{P}^1$ is locally injective at a with respect to the weak topology if $m_\varphi(a) = 1$. The converse holds if φ is separable.*
3. *If $\psi(z)$ is another nonconstant rational function, then*

$$m_{\psi \circ \varphi}(x) = m_\psi(\varphi(x)) \cdot m_\varphi(x) \quad \text{for all } x \in \mathbf{P}^1.$$

Remark 3.5. Statements (1) and (2) are also true for the strong topology since it is finer than the weak topology.

Remark 3.6. Part (2) of the proposition is proved in [2] under the hypothesis that the characteristic of k is zero, but their proof applies *mutatis mutandis* if φ is separable. See also [7, §2].

Corollary 3.7. *Let $\varphi \in k(z)$ be a nonconstant rational function, and let $\sigma_1, \sigma_2 \in \text{PGL}_2(k)$. Set $\psi = \sigma_2 \circ \varphi \circ \sigma_1$. Then $\mathcal{R}_\psi = \sigma_1^{-1}(\mathcal{R}_\varphi)$.*

Proof. This result is an immediate consequence of part (3) of the proposition and the fact that automorphisms are unramified:

$$m_\psi(x) = m_{\sigma_2}(\varphi(\sigma_1(x))) \cdot m_\varphi(\sigma_1(x)) \cdot m_{\sigma_1}(x) = m_\varphi(\sigma_1(x)), \quad x \in \mathbf{P}^1.$$

□

The fact that \mathbf{P}^1 is a tree implies that a rational function is injective on each connected component of the complement of the ramification locus.

Corollary 3.8. *Let $\varphi \in k(z)$ be a nonconstant rational function, and let $U \subset \mathbf{P}^1$ be a connected weak open subset. If $\varphi|_U$ is not injective, then U contains a ramified point.*

Proof. Let x, y be arbitrary distinct points of U . The segment $[x, y]$ is contained in U by connectedness. If U does not contain a ramified point, then φ is locally injective at every point of $[x, y]$ (Proposition 3.4). In particular, the image path $[x, y] \rightarrow \varphi([x, y])$ cannot have any backtracking. As \mathbf{P}^1 contains no loop, it follows that $\varphi(x) \neq \varphi(y)$, so that φ is injective. □

3.2 The Directional Multiplicity

Essentially all of the ideas in this section are due to Rivera-Letelier[10, §4], although we will adhere to the notation and terminology of Baker and Rumely[2, §9.1].

Proposition 3.9 ([2, pp.261–266]). *Let $\varphi \in k(z)$ be a nonconstant rational function, let $x \in \mathbf{P}^1$, and let $\vec{v} \in T_x$. Then there is a positive integer m and a point $x' \in \mathcal{B}_x(\vec{v})^-$ satisfying the following:*

1. *$m_\varphi(y) = m$ for all $y \in (x, x')$, and*
2. *$\rho(\varphi(x), \varphi(y)) = m \cdot \rho(x, y)$ for all $y \in (x, x')$.*

The integer m in the proposition is called the **directional multiplicity**, and we denote it by $m_\varphi(x, \vec{v})$. Part (1) shows that it satisfies $m_\varphi(x, \vec{v}) \leq \deg(\varphi)$.

For the next statement, a **generalized open Berkovich disk** is a weakly open set of the form $\mathcal{B}_x(\vec{v})^-$ for some point $x \in \mathbf{P}^1$ and some tangent vector \vec{v} at x . Equivalently, a weak open subset is a generalized open Berkovich disk if and only if it has exactly one boundary point. The following fundamental result is due to Rivera-Letelier [10, §4.1].

Proposition 3.10 ([2, Prop. 9.41]). *Let $\varphi \in k(z)$ be a nonconstant rational function. Let $\mathcal{B} = \mathcal{B}_x(\vec{v})^-$ be a generalized open Berkovich disk. Then $\varphi(\mathcal{B})$ always contains the generalized open Berkovich disk $\mathcal{B}' = \mathcal{B}_{\varphi(x)}(\varphi_*(\vec{v}))^-$, and either $\varphi(\mathcal{B}) = \mathcal{B}'$ or $\varphi(\mathcal{B}) = \mathbf{P}^1$. Set $m = m_\varphi(x, \vec{v})$ for the directional multiplicity.*

1. *If $\varphi(\mathcal{B}) = \mathcal{B}'$, then for each $y \in \mathcal{B}'$ there are exactly m solutions to $\varphi(z) = y$ in \mathcal{B} (counted with multiplicities).*
2. *If $\varphi(\mathcal{B}) = \mathbf{P}^1$, then there is a unique integer $s > 0$ such that for each $y \in \mathcal{B}'$, there are $s + m$ solutions to $\varphi(z) = y$ in \mathcal{B} (counted with multiplicities), and for each $y \in \mathbf{P}^1 \setminus \overline{\mathcal{B}'}$ there are s solutions to $\varphi(z) = y$ in \mathcal{B} (counted with multiplicities).*

The previous result shows $\varphi(\mathcal{B})$ is a generalized open Berkovich disk or else it is \mathbf{P}^1 . The next gives a useful criterion for determining which of these cases occurs.

Proposition 3.11 ([2, Thm. 9.42]). *Let $\varphi \in k(z)$ be a nonconstant rational function, and let \mathcal{B} be a generalized open Berkovich disk with boundary point ζ . Then $\varphi(\mathcal{B})$ is a generalized open Berkovich disk if and only if for each $c \in \mathcal{B}$, the multiplicity function m_φ is nonincreasing on the directed segment $[\zeta, c]$.*

The next result gives an algebraic relationship between the multiplicity $m_\varphi(x)$ and the directional multiplicities $m_\varphi(x, \vec{v})$ for $\vec{v} \in T_x$.

Proposition 3.12 ([2, Thm. 9.22]). *Let $\varphi \in k(z)$ be a nonconstant rational function and let $x \in \mathbf{P}^1$.*

1. *(Directional Multiplicity Formula) For each tangent vector \vec{w} at $\varphi(x)$, we have*

$$m_\varphi(x) = \sum_{\substack{\vec{v} \in T_x \\ \varphi_*(\vec{v}) = \vec{w}}} m_\varphi(x, \vec{v}).$$

2. *The induced map $\varphi_* : T_x \rightarrow T_{\varphi(x)}$ is surjective. If x is of type I, III, or IV, then $m_\varphi(x) = m_\varphi(x, \vec{v})$ for each tangent vector $\vec{v} \in T_x$.*

Corollary 3.13. *Let $\varphi \in k(z)$ be a nonconstant rational function. Then \mathcal{R}_φ is a closed subset of \mathbf{P}^1 with no isolated point (for both the weak and strong topologies).*

Proof. Proposition 3.4(1) immediately implies that \mathcal{R}_φ is closed. Proposition 3.12(1) shows that if $m_\varphi(x) > 1$, then there is a direction $\vec{v} \in T_x$ such that $m_\varphi(x, \vec{v}) > 1$. It follows from Proposition 3.9(1) that there exists $x' \in \mathcal{B}_x(\vec{v})^-$ such that $m_\varphi(y) = m_\varphi(x, \vec{v}) > 1$ for all $y \in (x, x')$. Hence \mathcal{R}_φ has no isolated point. \square

The following proposition, due to Rivera-Letelier, gives the best technique for determining the value of the multiplicity function at a type II point.

Proposition 3.14 (Algebraic Reduction Formula, [2, Thm. 9.42]). *Let $\varphi \in k(z)$ be a nonconstant rational function, and let $x \in \mathbf{P}^1$ be a point of type II. Put $y = \varphi(x)$, choose $\sigma_1, \sigma_2 \in \mathrm{PGL}_2(k)$ such that $\sigma_1(x) = \sigma_2(y) = \zeta_{0,1}$, and set $\psi(z) = \sigma_2 \circ \varphi \circ \sigma_1^{-1}$. Then ψ has nonconstant reduction $\tilde{\psi}$ and*

$$m_\varphi(x) = \deg(\tilde{\psi}).$$

For each $a \in \mathbb{P}^1(\tilde{k})$, if $\vec{v}_a \in T_x$ is the associated tangent direction under the bijection between T_x and $\mathbb{P}^1(\tilde{k})$ afforded by $(\sigma_1)_$, we have*

$$m_\varphi(x, \vec{v}_a) = m_{\tilde{\psi}}(a).$$

As an application of the results in this section, we describe the ramification locus for inseparable rational functions.

Proposition 3.15. *Suppose k has characteristic $p > 0$ (and hence residue characteristic p , in accordance with our conventions). Let $\varphi \in k(z)$ be a nonconstant inseparable rational function. Then $\mathcal{R}_\varphi = \mathbf{P}^1$.*

Proof. We begin by showing that $m_F \equiv p$, where $F \in k(z)$ is the relative Frobenius map defined by $F(z) = z^p$. For a closed disk $D(a, r)$ with rational radius and any $b \in D(a, r)$ with $|a - b| = r$, observe that

$$\psi(z) = b^{-p}[F(bz + a) - F(a)] = F(z).$$

Hence $m_F(\zeta_{a,r}) = m_F(\zeta_{0,1}) = p$ by the Algebraic Reduction Formula. Since m_F takes the same value at any type II point, we conclude that $m_F \equiv p$ (Proposition 3.9(1)).

Now we may factor φ uniquely as $\varphi = \psi \circ F^\ell$, where $\psi \in k(z)$ is separable, $\ell \geq 1$, and $F^\ell = F \circ \cdots \circ F$ is the ℓ -fold iterate of F . By Proposition 3.4(3), we see that

$$\begin{aligned} m_\varphi(x) &= m_\psi\left(F^\ell(x)\right) \cdot m_F\left(F^{\ell-1}(x)\right) \cdot m_F\left(F^{\ell-2}(x)\right) \cdots m_F(x) \\ &= p^\ell \cdot m_\psi\left(F^\ell(x)\right) \geq p^\ell. \end{aligned}$$

As x is arbitrary, $\mathcal{R}_\varphi = \mathbf{P}^1$. □

3.3 The Surplus Multiplicity

With the notation in Proposition 3.10, we define the **surplus multiplicity** $s_\varphi(\mathcal{B})$ to be zero if $\varphi(\mathcal{B})$ is a generalized open Berkovich disk, and to be $s_\varphi(\mathcal{B}) = s$ if $\varphi(\mathcal{B}) = \mathbf{P}^1$. As $\mathcal{B} = \mathcal{B}_x(\vec{v})^-$, we will also write $s_\varphi(x, \vec{v}) = s_\varphi(\mathcal{B})$. The intuition behind the terminology “surplus multiplicity” is that for $y \in \mathcal{B}'$, there are always at least m solutions to $\varphi(\zeta) = y$ with $\zeta \in \mathcal{B}$, and there are $s_\varphi(\mathcal{B})$ “extra” solutions depending on the nature of φ and \mathcal{B} .

The surplus multiplicity gives a lower bound for the number of pre-images of a given point inside certain open Berkovich disks. This fact is extremely important for bounding the number of connected components of \mathcal{R}_φ .

Proposition 3.16. *Let $\varphi \in k(z)$ be a nonconstant rational function, let $x \in \mathbf{P}^1$ be a type II point, and let $\vec{v} \in T_x$ be a tangent direction. If $\mathcal{B} = \mathcal{B}_x(\vec{v})^-$, then for every $y \in \mathbf{P}^1$,*

$$\#\{\zeta \in \mathcal{B} : \varphi(\zeta) = y\} \geq s_\varphi(\mathcal{B}).$$

Before starting the proof, we give an alternate description of the surplus multiplicity at the Gauss point. Let $\varphi = f/g$ be normalized. The surplus multiplicity is invariant under postcomposition by an element of $\mathrm{PGL}_2(k)$, so it suffices to assume that $\varphi(\zeta_{0,1}) = \zeta_{0,1}$, in which case φ has nonconstant reduction. In particular, this means that each of f and g has a coefficient with absolute value 1.

Write $F(X, Y) = Y^{\deg(\varphi)} f(X/Y)$ and $G(X, Y) = Y^{\deg(\varphi)} g(X/Y)$ for the homogenizations of f and g . Write \tilde{F} and \tilde{G} for the reductions of F and G , respectively; these reductions are nonzero since f and g each has a coefficient with absolute value 1. Let $H = \gcd(\tilde{F}, \tilde{G}) \in \tilde{k}[X, Y]$; it exists since \tilde{F} and \tilde{G} are homogeneous, and it is unique up to multiplication by a nonzero element of the residue field.

Now let $a \in \mathbb{P}^1(\tilde{k})$, and write $\mathcal{B} = \mathcal{B}_{\zeta_{0,1}}(\vec{a})^-$ for the corresponding open Berkovich disk. We claim that the surplus multiplicity of \mathcal{B} is equal to the multiplicity of a as a root of H . To see it, change coordinates on the source and target by an element of $\mathrm{PGL}_2(k^\circ)$ so that $\vec{a} = \varphi_*(\vec{a}) = \vec{0}$. The induced map $T_{\zeta_{0,1}} \rightarrow T_{\zeta_{0,1}}$ on sets of tangent vectors is given in homogeneous coordinates by

$$\varphi_* = \tilde{\varphi} = \left(\frac{\tilde{F}}{H} : \frac{\tilde{G}}{H} \right).$$

Since φ_* maps $\vec{0}$ to $\vec{0}$ with multiplicity $m = m_\varphi(\zeta_{0,1}, \vec{0})$, we see that $X^m \parallel \tilde{F}/H$. Let $S \geq 0$ be defined by $X^S \parallel H$. It follows that X^{m+S} evenly divides \tilde{F} , or equivalently that F has $m + S$ zeros in the disk $D(0, 1)^-$ (counted with multiplicity). In fact, this same conclusion holds with zero replaced by any $y \in D(0, 1)^-$, which shows that $S = s_\varphi(\zeta_{0,1}, \vec{0})$ is the surplus multiplicity of the disk $\mathcal{D}(0, 1)^-$. We summarize this conclusion as

Lemma 3.17. *Let $\varphi = f/g \in k(z)$ be a nonconstant normalized rational function with nonconstant reduction. Set $F(X, Y) = Y^{\deg(\varphi)} f(X/Y)$ and set $G(X, Y) = Y^{\deg(\varphi)} g(X/Y)$ for the homogenizations of f and g , and let $H = \gcd(\tilde{F}, \tilde{G})$ be a greatest common divisor of their reductions. For each $a \in \mathbb{P}^1(\tilde{k})$, the surplus multiplicity of the disk $\mathcal{B}_{\zeta_{0,1}}(\vec{a})^-$ is equal to the multiplicity of a as a root of H .*

If $x \in \mathbf{P}^1$ is a type II point and $\varphi \in k(z)$ is any nonconstant rational function, then an immediate consequence of this characterization of the surplus multiplicity and the Algebraic Reduction Formula is the following:

$$m_\varphi(x) + \sum_{\vec{v} \in T_x} s_\varphi(x, \vec{v}) = \deg(\varphi). \quad (3.1)$$

And while we will not need it in what follows, this formula actually holds at any $x \in \mathbf{P}^1$. The proof is trivial for points of type I or type IV since there is only one tangent direction to consider, and one can use Corollary 4.3 below to reduce the type III case to the type II case.

Proof of Proposition 3.16. Write $\mathcal{B}' = \mathcal{B}_{\varphi(x)}(\varphi_*(\vec{v}))^-$. If $y \neq \varphi(x)$, then $y \in \mathcal{B}'$ or $y \in \mathbf{P}^1 \setminus \overline{\mathcal{B}'}$. Proposition 3.10 immediately gives the desired conclusion. So we may assume that $y = \varphi(x)$ for the remainder of the proof. We are going to construct an open Berkovich disk $\mathcal{B}_0 \subsetneq \mathcal{B}$ such that $\mathcal{B}'_0 \subsetneq \mathcal{B}'$ and $s = s_\varphi(\mathcal{B}_0) = s_\varphi(\mathcal{B})$. Since $y \notin \mathcal{B}'_0$, we find that

$$\#\{\zeta \in \mathcal{B} : \varphi(\zeta) = y\} \geq \#\{\zeta \in \mathcal{B}_0 : \varphi(\zeta) = y\} = s,$$

which will complete the proof.

Without loss of generality, we may change coordinate on the source and target so that $x = \varphi(x) = \zeta_{0,1}$ and so that $\vec{v} = \varphi_*(\vec{v}) = \vec{0}$. Then $\mathcal{B} = \mathcal{D}(0,1)^- = \mathcal{B}'$. Write $m = m_\varphi(\zeta_{0,1}, \vec{0})$, and for $t \in k^\circ \setminus \{0\}$, define

$$\varphi_t(z) = t^{-m} \varphi(tz).$$

For $|t|$ sufficiently close to 1, once φ_t is properly normalized we will see that it has reduction $\tilde{\varphi}_t(z) = cz^m$ for some nonzero $c \in \tilde{k}$, and that $s_{\varphi_t}(\mathcal{D}(0,1)^-) = s$. In terms of our original coordinate, this translates into $s_\varphi(\mathcal{D}(0,|t|)^-) = s$. Then the disks $\mathcal{B}_0 = \mathcal{D}(0,|t|)^-$ and $\mathcal{B}'_0 = \mathcal{D}(0,|t|^m)^-$ are the ones we seek.

We begin by writing φ in normalized form as

$$\varphi(z) = \frac{a_d z^d + \cdots + a_0}{b_d z^d + \cdots + b_0},$$

with $a_i, b_j \in k^\circ$ and some coefficient in the numerator and denominator having absolute value 1. Let $s = s_\varphi(\zeta_{0,1}, \vec{0})$ be the associated surplus multiplicity. The Algebraic Reduction Formula and Lemma 3.17 shows that $|a_{m+s}| = |b_s| = 1$, and that $|a_i| < 1$ for $i < m+s$ and that $|b_j| < 1$ for $j < s$. Now observe that

$$\begin{aligned} \varphi_t(z) &= \frac{t^{-m-s}}{t^{-s}} \cdot \varphi(tz) \\ &= \frac{a_d t^{d-m-s} z^d + \cdots + a_{m+s} z^{m+s} + \cdots + t^{-m-s} a_0}{b_d t^{d-s} z^d + \cdots + b_s z^s + \cdots + t^{-s} b_0}. \end{aligned} \tag{3.2}$$

Define r_0 to be the maximum element of the set

$$\{|a_i|^{1/(m+s-i)} : i = 0, \dots, m+s-1\} \cup \{|b_j|^{1/(s-j)} : j = 0, \dots, s-1\}.$$

If we assume that $r_0 < |t| < 1$, then

$$|a_i t^{i-m-s}| \begin{cases} < 1 & \text{if } i \neq m+s \\ = 1 & \text{if } i = m+s \end{cases}, \quad |b_j t^{j-s}| \begin{cases} < 1 & \text{if } j \neq s \\ = 1 & \text{if } j = s \end{cases}. \tag{3.3}$$

Thus the presentation of φ_t given in (3.2) is normalized, and its reduction is given by $\tilde{\varphi}_t(z) = (\tilde{a}_{m+s}/\tilde{b}_s)z^m$. Moreover, the calculation (3.3) shows that a factor of z^s exactly cancels from the numerator and denominator when we reduce modulo k° , which is the same as saying that $s_{\varphi_t}(\mathcal{D}(0,1)^-) = s$ (Lemma 3.17). \square

The surplus multiplicity of a disk is closely tied to the number of critical points contained within it. The following result is the key to bounding the number of connected components of the ramification locus.

Proposition 3.18. *Let $\varphi \in k(z)$ be a nonconstant rational function. Suppose $x \in \mathbf{P}^1$ is a type II point and $\vec{v} \in T_x$ is a tangent direction such that $m_\varphi(x, \vec{v}) = 1$. Then we have the equality*

$$\sum_{c \in \mathcal{B}_x(\vec{v})^-} w_\varphi(c) = 2s_\varphi(x, \vec{v}).$$

(The sum is actually finite, being over the critical points in $\mathcal{B}_x(\vec{v})^-$.)

Proof. Change coordinates on the source and target so that $x = \varphi(x) = \zeta_{0,1}$ and $\mathcal{B}_x(\vec{v})^- = \mathcal{D}(0, 1)^-$. Note that φ must be separable, else its reduction $\tilde{\varphi}$ will be inseparable, so that $m_\varphi(x, \vec{v}) \geq p$ by the Algebraic Reduction Formula. In particular, φ has only finitely many critical points, so there are only finitely many connected components of $\mathbf{P}^1 \setminus \{\zeta_{0,1}\}$ that contain one. After a further change of coordinate on the source if necessary, we may assume that no critical point lies in the open Berkovich disk $\mathcal{B}_{\zeta_{0,1}}(\infty)^-$; equivalently, each critical point has absolute value at most 1.

We may suppose $\varphi = f/g$ is normalized, and set $h = \gcd(\tilde{f}, \tilde{g})$ with h monic. Write $\tilde{f} = hf_1$ and $\tilde{g} = hg_1$ for some $f_1, g_1 \in \tilde{k}[z]$. Since $m_\varphi(\zeta_{0,1}, \vec{0}) = 1$ by hypothesis, we see that f_1 has a simple zero at the origin and $g_1(0) \neq 0$. Hence

$$\text{ord}_{z=0} \text{Wr}_{\tilde{\varphi}} = \text{ord}_{z=0}(f_1'g_1 - f_1g_1') = 0.$$

Since $\varphi = f/g$ is normalized, we see that $\text{Wr}_\varphi \in k^\circ[z]$, and we may compute

$$\begin{aligned} \widetilde{\text{Wr}}_\varphi &= \tilde{f}'\tilde{g} - \tilde{f}\tilde{g}' \\ &= (h'f_1 + hf_1')hg_1 - hf_1(hg_1' + h'g_1) \\ &= \text{Wr}_{\tilde{\varphi}} \cdot h^2. \end{aligned}$$

With our choice of coordinates, all of the roots of Wr_φ have absolute value at most 1, and hence

$$\begin{aligned} \sum_{c \in \mathcal{D}(0,1)^-} w_\varphi(c) &= \text{ord}_{z=0} \widetilde{\text{Wr}}_\varphi \\ &= 2 \text{ord}_{z=0}(h) = 2s_\varphi(\zeta_{0,1}, \vec{0}), \end{aligned}$$

where the final equality follows upon dehomogenizing Lemma 3.17. \square

4 Extension of Scalars

Let K/k be an extension of algebraically closed and complete non-Archimedean fields, where as usual we assume the absolute value on k is nontrivial (and hence also on K). To distinguish between objects defined over k and those over K , we will decorate our notation with subscripts. For example, $D_k(0, 1)$ and \mathbf{P}_k^1 will denote the classical closed unit disk and the Berkovich projective line defined over k , respectively.

We will be occupied for most of this section with the proof of the following result.

Theorem 4.1. *To each extension K/k of algebraically closed and complete non-Archimedean fields, there exists a canonical inclusion map $\iota_k^K : \mathbf{P}_k^1 \rightarrow \mathbf{P}_K^1$ with the following properties:*

1. $\iota_k^K(\zeta_{k,\mathbf{a},\mathbf{r}}) = \zeta_{K,\mathbf{a},\mathbf{r}}$ for each decreasing sequence of closed disks $D_k(\mathbf{a}, \mathbf{r}) = (D(a_i, r_i))_{i \geq 0}$ with $a_i \in k$ and $r_i \in \mathbb{R}_{\geq 0}$. In particular, ι_k^K extends the natural inclusion $\mathbb{P}^1(k) \hookrightarrow \mathbb{P}^1(K)$ on classical points.
2. If K'/K is a further extension, then $\iota_k^{K'} = \iota_K^{K'} \circ \iota_k^K$.
3. ι_k^K is continuous for the weak topologies on \mathbf{P}_k^1 and \mathbf{P}_K^1 . In particular, $\iota_k^K(\mathbf{P}_k^1)$ is a compact subset of \mathbf{P}_K^1 for the weak topology.

4. ι_k^K is an isometry for the path-distance metric ρ .

5. Write $\iota = \iota_k^K$. For each $x \in \mathbf{P}_k^1$, there exists an injective map $\iota_* : T_x \rightarrow T_{\iota(x)}$ such that property that $\iota(\mathcal{B}_x(\vec{v})^-) \subset \mathcal{B}_{\iota(x)}(\iota_*(\vec{v}))^-$ for every $\vec{v} \in T_x$.

Remark 4.2. The map ι_k^K in the theorem is not a morphism of k -analytic spaces except in the case $k = K$, and so we cannot simply appeal to general principles in analytic geometry to determine its properties. Indeed, if it were k -analytic, then its construction below would imply the existence of a k -analytic morphism of Berkovich disks $\mathcal{D}_k(0, 1) \rightarrow \mathcal{D}_K(0, 1)$. Passing to rings of functions, there would exist a k -morphism of Tate algebras $K\{T\} \rightarrow k\{T\}$. The image of K must lie in a subfield of $k\{T\}$ containing k , and so it must be k itself.

An important consequence of the continuity properties of the map ι_k^K is the following application to multiplicities of rational functions.

Corollary 4.3. *Let K/k be an extension of algebraically closed and complete non-Archimedean fields, let $\varphi \in k(z)$ be a nonconstant rational function, and let $\iota = \iota_k^K : \mathbf{P}_k^1 \rightarrow \mathbf{P}_K^1$ be the inclusion map from the theorem. The following assertions hold:*

1. If $\varphi_K \in K(z)$ is given by extension of scalars, then $\varphi_K \circ \iota = \iota \circ \varphi$.
2. For every $x \in \mathbf{P}_k^1$, we have $m_{\varphi_K}(\iota(x)) = m_{\varphi}(x)$. In particular, $\iota^{-1}(\mathcal{R}_{\varphi_K}) = \mathcal{R}_{\varphi}$.
3. For each $x \in \mathbf{P}_k^1$ and each $\vec{v} \in T_x$, we have

$$m_{\varphi_K}(\iota(x), \iota_*(\vec{v})) = m_{\varphi}(x, \vec{v}) \quad \text{and} \quad s_{\varphi_K}(\iota(x), \iota_*(\vec{v})) = s_{\varphi}(x, \vec{v}).$$

Proof of Theorem 4.1. Define $\iota_k^K(\zeta_{k,\mathbf{a},\mathbf{r}}) = \zeta_{K,\mathbf{a},\mathbf{r}}$ and $\iota_k^K(\infty) = \infty$. Since cofinality of nested sequences of disks is preserved under base extension, this map is evidently well-defined and injective. Compatibility of the family of maps ι_k^\bullet is clear from the definition.

For the remainder of the proof, we assume that the extension K/k is fixed and write $\iota = \iota_k^K$ for simplicity. To prove weak continuity of ι , we observe that it suffices to prove $\iota^{-1}(\mathcal{D}(a, r)^-)$ is open and $\iota^{-1}(\mathcal{D}(a, r))$ is closed for every $a \in K$ and $r \in \mathbb{R}_{\geq 0}$. The proof breaks naturally into several cases.

Case 1: $D_K(a, r)^- \cap k \neq \emptyset$. Without loss of generality, we may assume that $a \in k$. We will now show that $\iota^{-1}(\mathcal{D}_K(a, r)^-) = \mathcal{D}_k(a, r)^-$. Suppose first that $\zeta_{k,\mathbf{b},\mathbf{s}} \in \iota^{-1}(\mathcal{D}_K(a, r)^-)$. We see that

$$\begin{aligned} |(T - a)(\zeta_{k,\mathbf{b},\mathbf{s}})| &= \lim_{i \rightarrow \infty} \sup_{x \in D_k(b_i, s_i)} |x - a| \\ &\leq \lim_{i \rightarrow \infty} \sup_{x \in D_K(b_i, s_i)} |x - a| = |(T - a)(\zeta_{K,\mathbf{b},\mathbf{s}})| < r. \end{aligned}$$

Hence $\zeta_{k,\mathbf{b},\mathbf{s}} \in \mathcal{D}_k(a, r)^-$.

For the other containment, suppose that $\zeta_{k,\mathbf{b},\mathbf{s}} \in \mathcal{D}_k(a, r)^-$. Then $s_i < r$ and $|(T - a)(\zeta_{k,b_i,s_i})| < r$ for i sufficiently large; fix such an i for the moment. For arbitrary $x \in D_K(b_i, s_i)$ and $x' \in D_k(b_i, s_i)$, we see that

$$|x - a| = |(x - b_i) - (x' - b_i) + (x' - a)| \leq \max\{s_i, |(T - a)(\zeta_{k,b_i,s_i})|\} < r.$$

Taking the supremum over all $x \in D_K(b_i, s_i)$ shows $\zeta_{K, b_i, s_i} \in \mathcal{D}_K(a, r)^-$. Letting i tend to infinity, we see that $\zeta_{K, \mathbf{b}, \mathbf{s}} \in \mathcal{D}_K(a, r)^-$. (Note that $|(T - a)(\zeta_{K, b_i, s_i})|$ is by definition a nonincreasing sequence in the variable i .)

Case 2: $D_K(a, r) \cap k \neq \emptyset$. The argument here is virtually identical to the previous case. If we assume (as we may without loss) that $a \in k$, then $\iota^{-1}(\mathcal{D}_K(a, r)) = \mathcal{D}_k(a, r)$.

Case 3: $D_K(a, r)^- \cap k = \emptyset$. We will argue that $\iota^{-1}(\mathcal{D}_K(a, r)^-) = \emptyset$. Indeed, suppose to the contrary that there exists $\zeta_{k, \mathbf{b}, \mathbf{s}}$ such that $|(T - a)(\zeta_{K, \mathbf{b}, \mathbf{s}})| < r$. Then for i sufficiently large, we find that

$$|(T - a)(\zeta_{K, b_i, s_i})| = \sup_{x \in D_K(b_i, s_i)} |x - a| < r.$$

But then $b_i \in k \cap D_K(a, r)^-$, a contradiction.

Case 4: $D_K(a, r) \cap k = \emptyset$. Observe that

$$\mathcal{D}_K(a, r) = \{\zeta_{K, a, r}\} \cup \bigcup_{a' \in D_K(a, r)} \mathcal{D}_K(a', r)^-.$$

We have already shown that the pre-image of each of the latter sets is empty in Case 3, so that $\iota^{-1}(\mathcal{D}_K(a, r)) = \iota^{-1}(\zeta_{K, a, r})$. As ι is injective, we conclude that $\iota^{-1}(\mathcal{D}_K(a, r))$ is either empty or a single point. In either case, it is closed for the weak topology.

Next, ι is an isometry for the path-distance metric because it preserves affine diameters and because it is compatible with the partial orderings on \mathbf{P}_k^1 and \mathbf{P}_K^1 in the following sense: For every $x, x', y \in \mathbf{P}_k^1$, we have $x \preceq x' \Rightarrow \iota(x) \preceq \iota(x')$ and $\iota(x \vee y) = \iota(x) \vee \iota(y)$. Indeed, these observations are immediate from the definitions for points of types I, II, or III, and a limiting argument gives them for type IV points.

By continuity and injectivity, the image of the connected set $\mathcal{B}_x(\vec{v})^-$ under ι is connected and does not contain $\iota(x)$. So it must be contained in $\mathcal{B}_{\iota(x)}(\vec{w})^-$ for some $\vec{w} \in T_{\iota(x)}$. We define $\iota_*(\vec{v}) = \vec{w}$.

We must show that $\iota_* : T_x \rightarrow T_{\iota(x)}$ is injective. This is clear if x is of type I or type IV, since $\#T_x = 1$. So we now assume that x is of type II or III. Let $\vec{v}_1 \neq \vec{v}_2 \in T_x$. Choose $x_i \in \mathcal{B}_x(\vec{v}_i)^-$ for $i = 1, 2$. It suffices to show $\iota(x_1)$ and $\iota(x_2)$ lie in distinct connected components of $\mathbf{P}_K^1 \setminus \{\iota(x)\}$. If $\mathcal{B}_x(\vec{v}_2)^-$ contains ∞ , then $x_1 \prec x \prec x_2$. The ordering is compatible with ι , so that $\iota(x_1) \prec \iota(x) \prec \iota(x_2)$. This last inequality implies that $\iota(x_1)$ and $\iota(x_2)$ must lie in distinct connected components of $\mathbf{P}_K^1 \setminus \{\iota(x)\}$. By symmetry, we obtain the same conclusion if $\infty \in \mathcal{B}_x(\vec{v}_1)^-$. Finally, suppose that $\infty \notin \mathcal{B}_x(\vec{v}_i)^-$ for $i = 1, 2$. In that case, $x_i \prec x$ for $i = 1, 2$, and x_1 and x_2 are mutually incomparable under the partial ordering, and we have $x_1 \vee x_2 = x$. Then $\iota(x) = \iota(x_1) \vee \iota(x_2)$, which means $\iota(x_1)$ and $\iota(x_2)$ again lie in distinct connected components of $\mathbf{P}_K^1 \setminus \{\iota(x)\}$. \square

Proof of Corollary 4.3. The first assertion is trivial for type I points of \mathbf{P}_k^1 , and the full equality $\varphi_K \circ \iota = \iota \circ \varphi$ follows by weak continuity of all of the maps involved and the fact that type I points are dense in \mathbf{P}_k^1 .

For the second assertion, it evidently holds whenever x is a type I point by the algebraic description of the multiplicity in that case. Now let $x \in \mathbf{P}_k^1$ be arbitrary, and let V be a φ_K -saturated weak neighborhood of $\iota(x)$. Then the multiplicity $m = m_{\varphi_K}(\iota(x))$ is equal to $\#V \cap \varphi_K^{-1}(\{y\})$ for each $y \in \varphi_K(V) \cap \mathbb{P}^1(K)$ (Proposition 3.1). Now observe that $U = \iota^{-1}(V)$ is a φ -saturated weak neighborhood of x . Since φ is defined over the algebraically closed field k , we find that $\#U \cap \varphi^{-1}(\{y\}) = m$ for any $y \in \varphi(U) \cap \mathbb{P}^1(k)$. Thus $m_\varphi(x) = m$ as well.

Finally, let $x \in \mathbf{P}_k^1$, $\vec{v} \in T_x$. Write $\mathcal{B}_k = \mathcal{B}_x(\vec{v})^-$ and $\mathcal{B}_K = \mathcal{B}_{\iota(x)}(\iota_*(\vec{v}))^-$. Then the proof of the theorem shows $\iota^{-1}(\mathcal{B}_K) = \mathcal{B}_k$. The third assertion of the corollary now follows from

Propositions 3.9 and 3.10, the compatibility of ι and φ , and what we have already shown in the last paragraph. \square

5 The Locus of Inseparable Reduction

The notion of inseparable reduction at a type II point was first given by Rivera-Letelier. We spend this section extending his definition to an arbitrary point of \mathbf{P}^1 . Rivera-Letelier has shown that a type II point x lies in the strong interior of the ramification locus if and only if φ has inseparable reduction at x . In §7 we will show that this statement continues to hold for an arbitrary point $x \in \mathbf{P}^1$.

Let us begin by recalling Rivera-Letelier's definition. Let $\varphi \in k(z)$ be a nonconstant rational function and let $x \in \mathbf{P}^1$ be a type II point. Let $\sigma_1, \sigma_2 \in \mathrm{PGL}_2(k)$ be chosen so that $\sigma_1(\zeta_{0,1}) = x$ and $\sigma_2(\varphi(x)) = \zeta_{0,1}$. Then $\psi = \sigma_2 \circ \varphi \circ \sigma_1$ fixes the Gauss point, and so ψ has nonconstant reduction $\tilde{\psi}$. The reduction $\tilde{\psi}$ is well-defined up to pre- and post-composition with an element of $\mathrm{PGL}_2(\tilde{k})$. We say that φ has **inseparable reduction** at x if k has positive residue characteristic and $\psi \in \tilde{k}(z)$ is inseparable. We say that φ has **separable reduction** at x if it does not have inseparable reduction. This definition is stable under extension of scalars:

Proposition 5.1. *Let K/k be an extension of complete and algebraically closed non-Archimedean fields, and let $\iota_k^K : \mathbf{P}_k^1 \hookrightarrow \mathbf{P}_K^1$ be the canonical inclusion. Then ι maps type II points to type II points, and the function φ has inseparable reduction at a type II point $x \in \mathbf{P}_k^1$ if and only if φ_K has inseparable reduction at $\iota_k^K(x)$.*

Proof. If $x = \zeta_{k,a,r}$ is a type II point, then $r \in |k^\times| \subset |K^\times|$, and hence $\iota_k^K(x) = \zeta_{K,a,r}$ is also a type II point. We may suppose that $x = \zeta_{k,0,1} = \varphi(\zeta_{k,0,1})$ after a change of coordinate on the source and target. Note that ι_k^K is compatible with these changes of coordinate (Corollary 4.3). Since $\tilde{\varphi}_K = \tilde{\varphi}_{\tilde{K}}$, we see that φ has inseparable reduction at the Gauss point of \mathbf{P}_k^1 if and only if φ_K has inseparable reduction at the Gauss point of \mathbf{P}_K^1 . \square

In order to generalize the definition of inseparable reduction, we will need to know there exist certain kinds of extensions of the field k . The following result is well-known, although its proof seems not to be.

Proposition 5.2. *There exists an algebraically closed and complete extension K/k with trivial residue extension such that K is spherically closed and $|K^\times| = \mathbb{R}_{>0}$. In particular, \mathbf{P}_K^1 has no point of type III or type IV.*

Proof. The construction of a universal field Ω_p lying over the algebraic closure of \mathbb{Q}_p given in [12, pp.137–140] applies *mutatis mutandis* to our setting. It gives an extension \hat{K}/k that is algebraically closed and complete, spherically closed, and has the desired value group. However, there is no control over the residue field of \hat{K} in this construction.

Let S be the set of intermediate extensions of \hat{K}/k with trivial residue extension. Then S is nonempty, and the union of a linearly ordered collection of elements of S is again an element of S . Zorn's lemma guarantees the existence of a maximal element K , which we claim satisfies the conclusion of the proposition.

Evidently K is complete, since otherwise its completion would be a strictly larger element of S . Next we show that $|K^\times| = \mathbb{R}_{>0}$. For otherwise, there exists $r \in \mathbb{R}_{>0} \setminus |K^\times|$. Let \mathcal{A}_r be the

generalized Tate algebra $K\{r^{-1}T\}$; it is the K -algebra of series $f = \sum_{i \geq 0} a_i T^i$ with K -coefficients such that $|a_i| r^i \rightarrow 0$ as $i \rightarrow \infty$. The norm on \mathcal{A}_r is $\|f\|_r = \sup_{i \geq 0} |a_i| r^i$. Then \mathcal{A}_r is a domain, and its fraction field K_r has residue field $\tilde{k} = \tilde{K}$ and value group generated by r and $|K^\times|$. Thus K_r contradicts the maximality of K .

Now let K'/K be a finite extension. Since the corresponding extension of residue fields is finite, and since $\tilde{K} = \tilde{k}$ is algebraically closed, we see that $\tilde{K}' = \tilde{K}$. Hence $K' = K$ by maximality.

Finally, the spherical closure of K is the maximal extension with the same residue field and value group as K . By maximality, we find K is itself spherically closed. \square

Definition 5.3. Fix a nonconstant rational function $\varphi \in k(z)$. We say that φ has inseparable reduction at a type I point if and only if φ is an inseparable rational function. We have already defined above what it means for φ to have inseparable reduction at a point of type II. If $x \in \mathbf{P}_k^1$ is a point of type III or type IV, then the preceding proposition shows there exists an extension K/k of algebraically closed and complete non-Archimedean fields such that $\iota_k^K(x)$ is a point of type II. We say that φ has inseparable reduction at x if φ_K has inseparable reduction at $\iota_k^K(x)$. This definition is independent of the choice of field K (Proposition 5.1).

The notion of inseparable reduction at a type I or type II point is evidently intrinsic to the field k by the above definitions. This is also true of type III points:

Proposition 5.4. Suppose k has residue characteristic $p > 0$. Let $\varphi \in k(z)$ be a nonconstant rational function, and let $x \in \mathbf{P}^1$ be a type III point. Then φ has inseparable reduction at x if and only if $p \mid m_\varphi(x)$.

Proof. Write $m = m_\varphi(x) = m_\varphi(x, \vec{v}_1) = m_\varphi(x, \vec{v}_2)$, where $T_x = \{\vec{v}_1, \vec{v}_2\}$. Let K be an algebraically closed and complete extension of k such that $x_K = \iota_k^K(x)$ is a type II point of \mathbf{P}_K^1 . Write $\vec{w}_i = (\iota_k^K)_*(\vec{v}_i)$ for $i = 1, 2$. Choose $\sigma_1 \in \mathrm{PGL}_2(K)$ so that

$$\sigma_1(\zeta_{K,0,1}) = x, \quad (\sigma_1)_*(\vec{0}) = \vec{w}_1, \quad (\sigma_1)_*(\vec{\infty}) = \vec{w}_2.$$

Next choose $\sigma_2 \in \mathrm{PGL}_2(K)$ so that

$$\sigma_2(\varphi_K(x)) = \zeta_{K,0,1}, \quad (\sigma_2)_*((\varphi_K)_*(\vec{w}_1)) = \vec{0}, \quad (\sigma_2)_*((\varphi_K)_*(\vec{w}_2)) = \vec{\infty}.$$

Then the map $\psi = \sigma_2 \circ \varphi_K \circ \sigma_1$ satisfies $\psi(\zeta_{K,0,1}) = \zeta_{K,0,1}$, $\psi_*(\vec{0}) = \vec{0}$, and $\psi_*(\vec{\infty}) = \vec{\infty}$. Then $m = m_\psi(\zeta_{K,0,1}) = m_\psi(\zeta_{K,0,1}, \vec{0}) = m_\psi(\zeta_{K,0,1}, \vec{\infty})$ (Corollary 4.3). The Algebraic Reduction Formula implies that $\psi(z) = az^m$ for some nonzero $a \in \tilde{K}$, and the proof is complete since $p \mid m$ if and only if ψ has inseparable reduction at $\zeta_{K,0,1}$ if and only if φ has inseparable reduction at x . \square

Remark 5.5. With a little more work, one can give a characterization of inseparable reduction that is intrinsic to the field k for any non-classical point. In the sequel [6], we introduce a continuous function $\tau_\varphi : \mathbf{H} \rightarrow \mathbb{R}_{\geq 0}$ — defined purely in terms of the coefficients of φ — in order to study the behavior of the ramification locus away from the connected hull of the critical points. It turns out that $\tau_\varphi(x) > 0$ if and only if φ has inseparable reduction at x .

6 Connected Components

We open this section by giving a bound on the number of connected components that the ramification locus may have (Theorem A). Then we study the part of the ramification locus lying off of the connected hull of the critical points. We also give sufficient conditions for when $\mathcal{R}_\varphi \subset \text{Hull}(\text{Crit}(\varphi))$. Finally, we show that — subject to the bound given by Theorem A — any number of connected components is achievable.

Proposition 6.1. *Let $\varphi \in k(z)$ be a nonconstant rational function. Let $x \in \mathbf{P}^1$ be a point with $m_\varphi(x) > 1$, and let X be the connected component of \mathcal{R}_φ containing x . Then X contains at least $2m_\varphi(x) - 2 \geq 2$ critical points of φ counted with weights.*

Proof of Theorem A. The proposition shows that each connected component of \mathcal{R}_φ contains at least two critical points, while the Hurwitz formula bounds the number of critical points of a separable rational function by $2\deg(\varphi) - 2$. Hence Theorem A follows in the separable case. Recall that if φ is inseparable, then $\mathcal{R}_\varphi = \mathbf{P}^1$ and Theorem A is trivial. \square

Remark 6.2. If the characteristic of the field k is positive, then it is possible to have a connected component of \mathcal{R}_φ containing only one critical point (when counted *without* weight). For example, this is the case for any polynomial function of the form $\varphi(z) = f(z^p) + az$, where $f \in k[z]$ is a nonconstant polynomial and $a \in k$ is nonzero.

When $\text{char}(k) = 0$, it appears that one can sharpen the above proposition to assert that each connected component contains at least 2 *distinct* critical points. We prove this when $p = 0$ or $p > \deg(\varphi)$ in Proposition 6.7 below.

Proof of Proposition 6.1. If φ is inseparable, then $\mathcal{R}_\varphi = \mathbf{P}^1 = X$, and φ has infinitely many critical points. So the result is trivial in this case.

Suppose now that φ is separable. Let $\{U_\alpha\}$ be the collection of connected components of $\mathbf{P}^1 \setminus X$. Note that each U_α is an open Berkovich disk with a type II endpoint x_α (Proposition 3.12). Let $\vec{v}_\alpha \in T_{x_\alpha}$ be the tangent direction such that $U_\alpha = \mathcal{B}_{x_\alpha}(\vec{v}_\alpha)^-$. Then $m_\varphi(x_\alpha, \vec{v}_\alpha) = 1$, since otherwise $U_\alpha \cap X$ would be nonempty. Let $y = \varphi(x)$. Then we have the estimate

$$\begin{aligned}
\deg(\varphi) &= \#\{\zeta \in \mathbf{P}^1 : \varphi(\zeta) = y\} \\
&\geq m_\varphi(x) + \sum_{\alpha} \#\{\zeta \in U_\alpha : \varphi(\zeta) = y\} \\
&\geq m_\varphi(x) + \sum_{\alpha} s_\varphi(U_\alpha) \quad \text{by Proposition 3.16} \\
&= m_\varphi(x) + \frac{1}{2} \sum_{\alpha} \sum_{c \in U_\alpha} w_\varphi(c) \quad \text{by Proposition 3.18} \\
&= m_\varphi(x) + \frac{1}{2} \sum_{c \in \mathbf{P}^1 \setminus X} w_\varphi(c) \\
&\implies \sum_{c \in \mathbf{P}^1 \setminus X} w_\varphi(c) \leq 2[\deg(\varphi) - m_\varphi(x)] \\
&\implies \sum_{c \in X} w_\varphi(c) \geq [2\deg(\varphi) - 2] - 2[\deg(\varphi) - m_\varphi(x)] = 2m_\varphi(x) - 2.
\end{aligned}$$

\square

Before describing the part of the ramification locus lying outside the connected hull of the critical points, we need a couple of technical lemmas.

Lemma 6.3. *Let $\varphi = f/g \in k(z)$ be a nonconstant rational function in normalized form with nonconstant reduction. The following are equivalent:*

1. $\widetilde{(\varphi')} = 0$
2. $\widetilde{\text{Wr}}_\varphi = 0$ (where Wr_φ is the Wronskian of $\varphi = f/g$)
3. φ has inseparable reduction at the Gauss point

Proof. The equivalence of the first two statements is immediate since Wr_φ is the numerator of φ' . Write $h = \gcd(\tilde{f}, \tilde{g})$, $f_1 = \tilde{f}/h$, and $g_1 = \tilde{g}/h$. Then

$$\widetilde{(\varphi')} = \frac{\tilde{f}'\tilde{g} - \tilde{f}\tilde{g}'}{\tilde{g}^2} = \frac{(f_1h' + f_1'h)g_1h - f_1h(g_1h' + g_1'h)}{g_1^2h^2} = \frac{f_1'g_1 - f_1g_1'}{g_1^2} = (\tilde{\varphi})'.$$

Hence we may write $\tilde{\varphi}'$ without ambiguity. The equivalence of (1) and (3) now follows from Proposition 2.3. \square

Lemma 6.4. *Let $\varphi \in k(z)$ be a nonconstant rational function satisfying the following two hypotheses:*

- φ is not injective on the classical disk $D(0, 1)^-$, and
- φ has no critical point in the classical disk $D(0, 1)^-$.

Then $0 < p \leq \deg(\varphi)$ and φ has inseparable reduction at the Gauss point.

Remark 6.5. The proof of the lemma shows the following interesting result. If $p \nmid m_\varphi(\zeta_{0,1}, \vec{0})$, then $D(0, 1)^-$ contains a critical point, and there are at least $2s_\varphi(\zeta_{0,1}, \vec{0}) + m_\varphi(\zeta_{0,1}, \vec{0}) - 1$ critical points of φ in the closed disk $D(0, 1)$.

Proof. We may make a change of coordinates on the target so that $\varphi(\zeta_{0,1}) = \zeta_{0,1}$ and $\varphi_*(\vec{0}) = \vec{0}$. Write $\varphi = f/g$ in normalized form with

$$\begin{aligned} f(z) &= a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0, \\ g(z) &= b_d z^d + b_{d-1} z^{d-1} + \cdots + b_0, \end{aligned}$$

where $a_i, b_j \in k^\circ$. Assume a_d or b_d is nonzero. Write $m = m_\varphi(\zeta_{0,1}, \vec{0})$ and $s = s_\varphi(\zeta_{0,1}, \vec{0})$. The proof of Lemma 3.17 shows $z^{m+s} \parallel \tilde{f}$ and $z^s \parallel \tilde{g}$. Equivalently, we have

$$\begin{aligned} |a_i| &< 1 \text{ for } 0 \leq i \leq m+s-1 \text{ and } |a_{m+s}| = 1; \\ |b_j| &< 1 \text{ for } 0 \leq j \leq s-1 \text{ and } |b_s| = 1. \end{aligned}$$

We will now show that the first segment of the Newton polygon of the Wronskian Wr_φ has negative slope if $p \nmid m$, which is equivalent to saying that $D(0, 1)^-$ contains a root of the Wronskian — i.e., a critical point of φ . Evidently this is a contradiction.

Write $\text{Wr}_\varphi(z) = \sum c_j z^j \in k^\circ[z]$. From (2.1) we see that the constant coefficient of Wr_φ is $c_0 = a_1 b_0 - a_0 b_1$. Since φ is not injective on $\mathcal{D}(0, 1)^-$, we find $s + m > 1$, so that both $a_0, a_1 \in k^{\circ\circ}$, which implies $|c_0| < 1$. We also see that the coefficient on the monomial z^{2s+m-1} is

$$c_{2s+m-1} = \sum_{n \neq m+s} (2n - 2s - m) a_n b_{2s+m-n} + m a_{s+m} b_s.$$

We know that $|a_n| < 1$ for $n < s + m$ and that $|b_{2s+m-n}| < 1$ for $2s + m - n < s$, or equivalently when $n > s + m$. So each of the terms in the above sum has absolute value strictly less than 1, while the final term has absolute value $|m|$. If $p \nmid m$, the final term has absolute value 1 and hence dominates the sum. This means the point $(2s + m - 1, 0)$ lies on the Newton polygon of Wr_φ (although it may not be a vertex). Hence the first segment of the Newton polygon of φ has negative slope.

Thus we conclude that $p \mid m \leq \deg(\varphi)$, which gives the desired bounds on the residue characteristic in the lemma. Finally, observe that if any coefficient c_ℓ has absolute value 1, then as above we deduce the existence of a critical point of φ in the disk $D(0, 1)^-$. Thus $|c_\ell| < 1$ for all $\ell \geq 0$. It follows that $\tilde{\psi}' = 0$, and an application of Lemma 6.3 completes the proof. \square

Proposition 6.6. *Let $\varphi \in k(z)$ be a nonconstant rational function. Let U be an open Berkovich disk disjoint from $\text{Hull}(\text{Crit}(\varphi))$ with type II boundary point x . Suppose $U \cap \mathcal{R}_\varphi$ is nonempty. Then the following assertions are true:*

1. $0 < p \leq \deg(\varphi)$;
2. φ has inseparable reduction at x ; and
3. $U \cap \mathcal{R}_\varphi$ is connected (for both the weak and strong topologies).

Proof. Change coordinates on the source and target so that $x = \varphi(x) = \zeta_{0,1}$ and $U = \mathcal{D}(0, 1)^-$. The assumption $U \cap \mathcal{R}_\varphi \neq \emptyset$ implies that φ is not injective on U . Also, U contains no critical point by hypothesis. Thus φ has inseparable reduction at x and $0 < p \leq \deg(\varphi)$ (Lemma 6.4).

If $U \cap \mathcal{R}_\varphi$ were disconnected, then we could choose a subdisk $U' \subset U$ that intersects \mathcal{R}_φ , but whose boundary point x' lies outside of \mathcal{R}_φ . The complement of \mathcal{R}_φ is open, so we may take x' of type II. The previous paragraph would apply to U' to show that φ has inseparable reduction at x' , so that $m_\varphi(x') \geq p$, a contradiction. Thus $U \cap \mathcal{R}_\varphi$ is connected. \square

Proposition 6.7. *Let $\varphi \in k(z)$ be a nonconstant rational function, and suppose that the residue characteristic of k satisfies $p = 0$ or $p > \deg(\varphi)$. Then the following statements hold:*

1. $\mathcal{R}_\varphi \subset \text{Hull}(\text{Crit}(\varphi))$
2. The endpoints of \mathcal{R}_φ are exactly the critical points of φ . In particular, each connected component contains at least two distinct critical points.
3. If φ has ℓ distinct critical points, then the number of connected components of \mathcal{R}_φ is at most $\ell/2$.

Proof. Note φ has at least 2 distinct critical points, so that $\text{Hull}(\text{Crit}(\varphi))$ is not reduced to a point. Let \mathcal{B} be a connected component of $\mathbf{P}^1 \setminus \text{Hull}(\text{Crit}(\varphi))$ with boundary point x . Then x is of type II. Proposition 6.6 implies that $\mathcal{B} \cap \mathcal{R}_\varphi$ is empty, else $0 < p \leq \deg(\varphi)$. Hence $\mathcal{R}_\varphi \subset \text{Hull}(\text{Crit}(\varphi))$.

Evidently the critical points of φ are endpoints in \mathbf{P}^1 , and also in \mathcal{R}_φ . There are no type IV points in $\text{Hull}(\text{Crit}(\varphi))$. A type III point x satisfies $m_\varphi(x) = m_\varphi(x, \vec{v})$ for each of the two tangent directions \vec{v} at x (Proposition 3.12(2)), and hence x cannot be an endpoint. Finally, suppose x is a type II endpoint of \mathcal{R}_φ . After a change of coordinates, we may suppose $x = \zeta_{0,1} = \varphi(\zeta_{0,1})$ and that the single ramified direction is $\vec{0}$. Our discussion on the Hurwitz formula in §2.3.1 shows that $\tilde{\varphi}$ must be wildly ramified at 0, and hence

$$p \mid m_\varphi(x, \vec{v}) \leq m_\varphi(x) \leq \deg(\varphi),$$

which contradicts our hypothesis.

Finally, we note that each connected component of \mathcal{R}_φ is a nontrivial tree, and any such object has at least two endpoints. As the endpoints of \mathcal{R}_φ are precisely the critical points, we see that

$$\#\{\text{connected components of } \mathcal{R}_\varphi\} \leq \frac{1}{2} \#\{\text{endpoints of } \mathcal{R}_\varphi\} = \frac{\ell}{2}.$$

□

We close this section by showing that Theorem A is optimal.

Proposition 6.8. *Let k be an algebraically closed field that is complete with respect to a nontrivial non-Archimedean absolute value. Fix integers $1 \leq n < d$. Then there exists a rational function $\varphi \in k(z)$ of degree d whose ramification locus \mathcal{R}_φ has precisely n connected components.*

Proof. For the case $n = 1$, let φ be a polynomial of degree d . Then $m_\varphi(\infty) = d$, and so the connected component X of \mathcal{R}_φ containing ∞ must contain all of the critical points of φ (Proposition 6.1). Any other connected component of \mathcal{R}_φ would need to contain a critical point, so we conclude that $X = \mathcal{R}_\varphi$. (See also Corollary 8.4.)

We assume for the remainder of the proof that $n \geq 2$. It will be convenient to set $\ell = n - 1$ and construct a rational function whose ramification locus has $\ell + 1$ connected components.

Begin by selecting a rational function $\psi = f/g \in k(z)$ with the following properties:

- ψ has degree $d - \ell$;
- $\tilde{\psi} \in \tilde{k}(z)$ is a separable rational function of degree $d - \ell$;
- ∞ is not a critical point for ψ ;
- $\psi = f/g$ is normalized (see §2.3.2); and
- f and g are monic of degree $d - \ell$.

The set of separable rational functions in $\tilde{k}(z)$ of degree $d - \ell$ with simple critical points and non-vanishing leading coefficient in numerator and denominator is a Zariski open subset of the space of all rational functions of degree $d - \ell$. Choose such a rational function and lift its coefficients to k° ; if necessary, change coordinate on the source so that ∞ is not a critical point. Scaling f and g and perhaps making a scalar change of coordinate on the target allows one to assume f, g are monic.

Now select elements $a_1, a_2, \dots, a_\ell \in k^\circ$ with distinct nonzero images in the residue field \tilde{k} . For each $i = 2, \dots, \ell$, choose $b_i \in k^\circ$ such that $0 < |a_i - b_i| < 1$. Choose $t \in k^{\circ\circ} \setminus \{0\}$. Now we may define a rational function $\varphi \in k(z)$ by

$$\varphi(z) = \frac{(z - a_1)(z - a_2) \cdots (z - a_\ell)}{(z - b_2) \cdots (z - b_\ell)} \psi(z/t).$$

Evidently the numerator and denominator of φ have degree d and $d - 1$, respectively. To show that φ has degree d , we must show that no root of the numerator of φ coincides with a root of the denominator. Write

$$\psi(z) = \frac{z^{d-\ell} + \alpha_{d-\ell-1}z^{d-\ell-1} + \cdots + \alpha_0}{z^{d-\ell} + \beta_{d-\ell-1}z^{d-\ell-1} + \cdots + \beta_0}.$$

Then

$$\psi(z/t) = \frac{z^{d-\ell} + t\alpha_{d-\ell-1}z^{d-\ell-1} + \cdots + t^{d-\ell}\alpha_0}{z^{d-\ell} + t\beta_{d-\ell-1}z^{d-\ell-1} + \cdots + t^{d-\ell}\beta_0}.$$

A Newton polygon argument shows that the zeros and poles of $\psi(z/t)$ all lie in $D(0, 1)^-$. The a_i 's and b_j 's all have absolute value 1, and $a_i \neq b_j$ for any i, j by construction. Hence φ has degree d .

The reduction of φ is $\tilde{\varphi}(z) = z - \tilde{a}_1$. The Algebraic Reduction Formula shows $m_\varphi(\zeta_{0,1}) = 1$, which means that each connected component of \mathcal{R}_φ lies inside a connected component of $\mathbf{P}^1 \setminus \{\zeta_{0,1}\}$. For each $i = 2, \dots, \ell$, let U_i be the connected component of $\mathbf{P}^1 \setminus \{\zeta_{0,1}\}$ containing a_i (and b_i). First observe that the surplus multiplicity is $s_\varphi(U_i) = 1$ (Proposition 3.17). So U_i contains exactly 2 critical points (counted with weights) for $i = 2, \dots, \ell$ (Proposition 3.18), and hence U_i contains a single connected component of \mathcal{R}_φ (Proposition 6.1). Set $U_1 = \mathcal{D}(0, 1)^-$. Then $s_\varphi(U_1) = d - \ell$, so that U_1 contains $2(d - \ell)$ critical points (counted with weights). It remains for us to show that U_1 contains exactly two connected components of \mathcal{R}_φ .

Define

$$\eta(z) = \varphi(tz) = \frac{(tz - a_1)(tz - a_2) \cdots (tz - a_\ell)}{(tz - b_2) \cdots (tz - b_\ell)} \psi(z).$$

Then $\tilde{\eta}(z) = (-\tilde{a}_1)\tilde{\psi}(z)$, which has degree $d - \ell$, and so $s_\eta(\zeta_{0,1}, \infty) = \ell$ (Lemma 3.17). The open Berkovich disk $\mathcal{B}_{\zeta_{0,|t|}}(\vec{v})^-$ contains 2ℓ critical points, where \vec{v} is the tangent vector corresponding to the connected component of $\mathbf{P}^1 \setminus \{\zeta_{0,|t|}\}$ containing ∞ . We have already accounted for $2(\ell - 1)$ of those critical points above, and so there must be two more critical points — and hence exactly one more component of \mathcal{R}_φ — in the open annulus $\{x \in \mathbf{A}^1 : |t| < |T(x)| < 1\}$.

The reduction of η shows $m_\varphi(\zeta_{0,|t|}) = d - \ell$ (Algebraic Reduction Formula). Proposition 6.1 shows the connected component of \mathcal{R}_φ containing $\zeta_{0,|t|}$ also contains at least $2(d - \ell) - 2$ critical points. We have accounted for $2(\ell - 1) + 2 = 2\ell$ critical points in the preceding paragraphs, and we have just located $2(d - \ell) - 2$ more. The Hurwitz formula shows we have now found all of the critical points, and hence all of the connected components of \mathcal{R}_φ . That is, \mathcal{R}_φ has $\ell + 1$ connected components. \square

7 Endpoints and Interior Points

Here we determine the interior and endpoints of \mathcal{R}_φ for both the weak and strong topologies. We already saw in Proposition 3.15 that $\mathcal{R}_\varphi = \mathbf{P}^1$ if φ is itself an inseparable rational function; here we show this is the only case in which the weak interior of \mathcal{R}_φ is nonempty. Then we characterize the endpoints of the ramification locus. We finish the section by showing that the strong interior of \mathcal{R}_φ coincides with the locus of inseparable reduction. (The definitions were chosen so that this statement holds even when φ is inseparable.)

Proposition 7.1. *The weak interior of the ramification locus of a separable nonconstant rational function is empty.*

Proof. Suppose there exists a rational function $\varphi \in k(z)$ such that the weak interior of its ramification locus is nonempty. Any weak open subset of \mathbf{P}^1 contains infinitely many points of type I, and the type I points of the ramification locus are precisely the critical points. Thus φ has infinitely many critical points, and hence it must be inseparable by the Hurwitz formula. \square

Lemma 7.2. *Suppose k has positive residue characteristic p , and suppose $\varphi \in k(z)$ is a nonconstant rational function with nonconstant reduction. Let \vec{v} be a tangent direction at the Gauss point of \mathbf{P}^1 , and write $m = m_\varphi(\zeta_{0,1}, \vec{v})$. Then $p \mid m$ if and only if there exists a point $x \in \mathcal{B}_{\zeta_{0,1}}(\vec{v})^-$ such that φ has inseparable reduction at each point of the segment $(\zeta_{0,1}, x)$.*

Proof. Without loss of generality, we may replace k with an algebraically closed and complete extension in order to assume that \mathbf{P}_k^1 has no point of type III or IV. (See §4 and §5.) Moreover, we may change coordinates on the source and target in order to assume that $\vec{v} = \varphi_*(\vec{v}) = \vec{0}$. Write $m = m_\varphi(\zeta_{0,1}, \vec{0})$, and for $t \in k^{\circ\circ} \setminus \{0\}$, define

$$\varphi_t(z) = t^{-m} \varphi(tz).$$

To prove the lemma, it suffices to show that once φ_t is properly normalized, it has reduction $\tilde{\varphi}_t(z) = cz^m$ for some nonzero $c \in \tilde{k}$ whenever $t \in k^{\circ\circ}$ has absolute value sufficiently close to 1. Indeed, if $p \mid m$, then this shows φ has inseparable reduction at $\zeta_{0,|t|}$. This exact calculation was carried out in the proof of Proposition 3.16. \square

Proposition 7.3 (Endpoints of \mathcal{R}_φ). *Let $\varphi \in k(z)$ be a nonconstant rational function, and suppose $x \in \mathcal{R}_\varphi$ is an endpoint of the ramification locus. Then x is of type I, II, or IV.*

1. *If x is of type I, then it is a critical point of φ .*
2. *If x is of type II or IV, then φ has inseparable reduction at every point of some nonempty segment $(x, y) \subset \mathcal{R}_\varphi$. In particular, $0 < p \leq \deg(\varphi)$.*

Proof. Suppose first that $x \in \mathcal{R}_\varphi$ is of type III, so that it has exactly two tangent directions \vec{v}_1 and \vec{v}_2 . The local degree satisfies $m_\varphi(x, \vec{v}_1) = m_\varphi(x, \vec{v}_2) > 1$ (Proposition 3.12(2)). Hence x cannot be an endpoint of \mathcal{R}_φ (Proposition 3.9(1)).

Now let $x \in \mathcal{R}_\varphi$ be of type I. Then $m_\varphi(x) > 1$ is the usual algebraic multiplicity, and hence x must be a critical point of φ .

Next suppose that x is a type II endpoint of \mathcal{R}_φ . After a change of coordinate on the source and target, we may suppose that $x = \zeta_{0,1} = \varphi(\zeta_{0,1})$. Then φ has nonconstant reduction at x . Since x is an endpoint, we see that $m_\varphi(x, \vec{v}) > 1$ for precisely one tangent direction \vec{v} . Corollary 2.5 shows that $\tilde{\varphi}$ must be wildly ramified in the direction \vec{v} , and hence

$$p \mid m_\varphi(x, \vec{v}) \leq m_\varphi(x) \leq \deg(\varphi),$$

Since $p \mid m_\varphi(x, \vec{v})$, the result follows upon applying the preceding lemma.

Now suppose that $x \in \mathcal{R}_\varphi$ is of type IV. Let y be the closest point to x in $\text{Hull}(\text{Crit}(\varphi))$; more precisely, if U is the connected component of $\mathbf{P}_k^1 \setminus \text{Hull}(\text{Crit}(\varphi))$ containing x , then y is the unique boundary point of U . Let K/k be an extension of algebraically closed and complete non-Archimedean fields so that \mathbf{P}_K^1 has no point of type III or IV. Write $x_K = \iota_k^K(x)$ and $y_K = \iota_k^K(y)$. Proposition 6.6 implies that φ_K has inseparable reduction at every (type II) point of the segment (x_K, y_K) . Hence φ has inseparable reduction at every point of the segment (x, y) . \square

Rivera-Letelier has characterized when a type II point lies in the strong interior of the ramification locus:

Proposition 7.4 ([11, Prop. 10.2]). *Let $\varphi \in k(z)$ be a nonconstant rational function and let $x \in \mathbf{P}^1$ be a type II point. Then φ has inseparable reduction at x if and only if there exists a strong neighborhood V of x such that $m_\varphi(y) \geq p$ for each $y \in V$.*

Remark 7.5. While the result in [11] is stated over \mathbb{C}_p , the proof is valid for an arbitrary non-Archimedean field (with residue characteristic $p > 0$). Note that the statement is vacuous if $\text{char}(\tilde{k}) = 0$; indeed, φ has separable reduction at all points by definition, and \mathcal{R}_φ has empty strong interior (Proposition 6.7).

Corollary 7.6. *Let $\varphi \in k(z)$ be a nonconstant rational function. If $\mathcal{R}_\varphi \not\subset \text{Hull}(\text{Crit}(\varphi))$, then $0 < p \leq \deg(\varphi)$. Moreover, if Y is a connected component of $\mathcal{R}_\varphi \setminus \text{Hull}(\text{Crit}(\varphi))$, then each point of \overline{Y} is either a strong interior point of \mathcal{R}_φ or an endpoint of \mathcal{R}_φ .*

Remark 7.7. As a subspace of \mathcal{R}_φ , the unique boundary point of Y will be of type II in general. However, if k has positive characteristic p , then it is possible for φ to have a single critical point (counted without weight), in which case $\text{Hull}(\text{Crit}(\varphi)) = \partial Y$ consists of a single point of type I. The statement of the corollary applies in either case.

Proof. The assertion about the residue characteristic of k follows immediately from Proposition 6.7. Suppose that $y \in \overline{Y}$. If y is of type I or IV, it is an endpoint of \mathbf{P}^1 , and hence also of \mathcal{R}_φ . If y is of type II, define $S \subset T_y$ to be the set of tangent directions \vec{v} such that $m_\varphi(y, \vec{v}) > 1$. Then S is nonempty since \mathcal{R}_φ has no isolated point. If $\#S = 1$, then y is an endpoint. Otherwise, $\#S \geq 2$, and there exists an open Berkovich disk U disjoint from $\text{Hull}(\text{Crit}(\varphi))$ with boundary point y such that $U \cap \mathcal{R}_\varphi \neq \emptyset$. Thus φ has inseparable reduction at y (Proposition 6.6), and so y is a strong interior point of \mathcal{R}_φ by the above proposition.

If $y = \zeta_{a,r}$ is of type III, then we will show it is an interior point of \mathcal{R}_φ . Let K/k be an extension of algebraically closed and complete non-Archimedean fields such that $r \in |K^\times|$, and write $y_K = \iota_k^K(y)$. Then y_K is a type II point of \mathbf{P}_K^1 that lies off of the connected hull of the critical points of φ_K . A type III point can never be an endpoint of the ramification locus; it follows that y_K is not an endpoint of \mathcal{R}_{φ_K} (Proposition 4.3). The argument in the previous paragraph applied to y_K and φ_K shows that y_K is a strong interior point of \mathcal{R}_{φ_K} . If $V \subset \mathcal{R}_{\varphi_K}$ is a strong open neighborhood of y_K , then $(\iota_k^K)^{-1}(V) \subset \mathcal{R}_\varphi$ is a strong open neighborhood of y (Theorem 4.1). \square

Lemma 7.8. *Suppose k has positive residue characteristic. Let $\varphi \in k(z)$ be such that $s_\varphi(\zeta_{0,1}, \vec{v}) = 0$ for all $\vec{v} \neq \infty$, and suppose further that $\tilde{\varphi}(z) = h(z^p) + cz$ for some nonconstant polynomial $h \in \tilde{k}[z]$ and some nonzero c . Fix $\delta > 0$. Then there exists $\varepsilon > 0$ such that $\zeta_{B,|A|} \notin \mathcal{R}_\varphi$ for any $A, B \in k$ satisfying*

$$0 < |A| < q_k^{-\delta} \quad \text{and} \quad 1 < |B| < q_k^\varepsilon.$$

Proof. Let $A, B \in k$ satisfy $0 < |A| < q_k^{-\delta}$ and $|B| > 1$. Set $\psi(z) = A^{-1}[\varphi(Az + B) - \varphi(B)]$. If $\varphi(z) = f(z)/g(z)$, then

$$\psi(z) = \frac{A^{-1}[f(Az + B) - f(B)]}{g(Az + B)} + \frac{A^{-1}f(B)[g(B) - g(Az + B)]}{g(B)g(Az + B)}. \quad (7.1)$$

We will show that the first term above reduces to a linear polynomial in $\tilde{k}[z]$, and that the second vanishes modulo $k^{\circ\circ}$, provided that $|B|$ is sufficiently close to 1. The Algebraic Reduction Formula then implies $m_\psi(\zeta_{0,1}) = 1 = m_\varphi(\zeta_{B,|A|})$, so that $\zeta_{B,|A|}$ is not in the ramification locus.

Write φ in normalized form as

$$\varphi(z) = \frac{a_d z^d + \cdots + a_0}{b_d z^d + \cdots + b_0} = \frac{f(z)}{g(z)}.$$

Let D be the degree of the polynomial h in the statement of the lemma. The hypotheses on the surplus multiplicity and on the reduction of φ are equivalent to saying $|b_j| < 1$ for $j = 1, \dots, d$, that $|a_i| < 1$ for $i > Dp$, that $|a_i| < 1$ for $1 < i < Dp$ such that $p \nmid i$, and that $|b_0| = 1 = |a_1| = |a_{Dp}|$.

In the remainder of the proof, we write β for any positive real function that tends to zero as $|B| \rightarrow 1$, independently of A . Note also that if $|A| < q_k^{-\delta}$, then A is uniformly bounded away from 1. Consider the quantity

$$X_j := A^{-1} [a_j(Az + B)^j - a_j B^j] = a_j \sum_{1 \leq i \leq j} \binom{j}{i} A^{i-1} B^{j-i} z^i.$$

We will show that $\tilde{X}_j = 0$ for $j \neq 1$ provided $|B|$ is sufficiently close to 1. If $j > Dp$ and $|B|$ is sufficiently close to 1, then $|a_j| < 1$ implies every coefficient of X_j is bounded by $|a_j|(1 + \beta) < 1$. If $1 < j < Dp$ and $p \nmid j$, then each coefficient of X_j is bounded by $|a_j|(1 + \beta) < 1$ for the same reason. If $1 < j \leq Dp$ and $p \mid j$, then

$$X_j = j a_j B^{j-1} z + a_j A \sum_{2 \leq i \leq j} \binom{j}{i} A^{i-2} B^{j-i} z^i.$$

The linear coefficient has absolute value bounded by $|p|(1 + \beta) < 1$ since $p \mid j$, and the remaining coefficients are bounded by $|A|(1 + \beta) < q_k^{-\delta}(1 + \beta)$. The remaining cases $j = 0$ and $j = 1$ are treated by observing that $X_0 = 0$ and $X_1 = a_1 z$.

Next observe that

$$g(Az + B) - b_0 = \sum_{1 \leq j \leq d} b_j (Az + B)^j.$$

Since $|b_j| < 1$ for all $j > 0$, we see that $\widetilde{g(Az + B)} = \tilde{b}_0$ provided $|B|$ is sufficiently close to 1. Hence

$$\frac{A^{-1}[f(Az + B) - f(B)]}{g(Az + B)} = \frac{\sum_{0 \leq j \leq d} X_j}{g(Az + B)} \equiv \frac{a_1}{b_0} z \pmod{k^{\circ\circ}}.$$

Thus the first term in (7.1) has the desired reduction.

For the second term in (7.1), we observe that $g(Az + B) = g(B) + A \cdot E(z)$, where $E \in k^{\circ\circ}$ is a polynomial whose coefficients are bounded by $(1 + \beta) \max\{|b_j| : j > 0\}$. Note also that $|f(B)| \leq 1 + \beta$. Since $\widetilde{g(B)} = \tilde{b}_0$, it follows that

$$\frac{A^{-1} f(B) [g(B) - g(Az + B)]}{g(B) g(Az + B)} = \frac{-f(B) E(z)}{g(B) [g(B) + A \cdot E(z)]} \equiv 0 \pmod{k^{\circ\circ}}.$$

We have now show that the second term in (7.1) has the desired reduction when $|B|$ is sufficiently close to 1, which completes the proof. \square

Proposition 7.9. *Let $\varphi \in k(z)$ be a nonconstant rational function, and let $x \in \mathbf{P}^1$. Then φ has inseparable reduction at x if and only if x is an interior point of \mathcal{R}_φ for the strong topology.*

Proof. First, suppose x is of type I. By definition, the function φ has inseparable reduction at x if and only if φ is itself inseparable. In the case that φ is inseparable, we have $\mathcal{R}_\varphi = \mathbf{P}^1$ (Proposition 3.15), so that every classical point is a strong interior point. If φ is separable, we must show that x fails to be a strong interior point. A strong open neighborhood of x contains infinitely many type I points. But the type I points of \mathcal{R}_φ are precisely the critical points, of which φ has only finitely many. So x cannot be a strong interior point.

Now we suppose that x is of type II, III, or IV, and that φ has inseparable reduction at x . Let K/k be an extension of algebraically closed and complete non-Archimedean fields such that \mathbf{P}_K^1 has only type I and type II points (Proposition 5.2). Write $\iota = \iota_k^K$. Then $\iota(x)$ is a type II point, and Proposition 7.4 shows that φ_K has inseparable reduction at $\iota(x)$ if and only if there exists a strong open neighborhood V of $\iota(x)$ contained inside \mathcal{R}_{φ_K} . By shrinking V if necessary, we may assume it contains no type I point. Set $U = \iota^{-1}(V)$. Theorem 4.1 and its corollary show that $U \subset \mathcal{R}_\varphi$ is a strong open neighborhood of x . That is, x is a strong interior point of \mathcal{R}_φ .

For the reverse implication, we assume that $x \in \mathbf{P}^1$ is a strong interior point of \mathcal{R}_φ and show that φ has inseparable reduction at x . This is clear by Proposition 7.4 if x is of type II. Suppose x is of type III. The multiplicity $m_\varphi(y)$ is constant with value $m = m_\varphi(x)$ for all type II points y lying on some segment beginning at x (Propositions 3.9 and 3.12). Now each such point y that is sufficiently close to x in the strong topology must lie in the strong interior of \mathcal{R}_φ . So φ has inseparable reduction at y ; hence $p \mid m_\varphi(y) = m_\varphi(x)$; hence φ has inseparable reduction at x (Proposition 5.4).

Finally, suppose x is a type IV point in the strong interior of \mathcal{R}_φ . Note that x does not lie on the connected hull of the critical points of φ . Let K/k be an extension of non-Archimedean fields as in the second paragraph. In particular, $x_K = \iota(x)$ is a type II point, so it must be either an endpoint or a strong interior point of \mathcal{R}_{φ_K} (Corollary 7.6). In the latter case, φ_K has inseparable reduction at x_K (Proposition 7.4), and so φ has inseparable reduction at x (by definition).

It remains to show that x_K cannot be an endpoint of the ramification locus of φ_K . Suppose to the contrary that it is an endpoint. Let $\vec{v} \in T_{x_K}$ be the unique tangent direction such that $m_{\varphi_K}(x_K, \vec{v}) > 1$. We may select $\sigma_1, \sigma_2 \in \text{PGL}_2(K)$ so that $\sigma_1^{-1}(x_K) = \zeta_{K,0,1} = \sigma_2(\varphi_K(x_K))$, and so that $(\sigma_1)_*^{-1}(\vec{v}) = \vec{\infty} = (\sigma_2)_*((\varphi_K)_*(\vec{v}))$. Set $\psi(z) = \sigma_2 \circ \varphi_K \circ \sigma_1$. Since x is a type IV point, $m_\varphi(x) = m_{\varphi_K}(x_K) = m_{\varphi_K}(x_K, \vec{v}) > 1$. So $\tilde{\psi} \in K(z)$ is a rational function that fixes ∞ , and the (algebraic) multiplicity at infinity equals the degree of $\tilde{\psi}$. Thus $\tilde{\psi}$ is a polynomial function. Moreover, $\tilde{\psi}$ has no finite critical point, and so its formal derivative must be a nonzero constant $c \in \tilde{K}$. We conclude that $\tilde{\psi}(z) = h(z^p) + cz$ for some nonconstant polynomial $h \in \tilde{K}[z]$. Observe further that $s_\psi(\zeta_{K,0,1}, \vec{w}) = 0$ for all $\vec{w} \neq \vec{\infty}$ since x_K is the image of a type IV point in \mathbf{P}_K^1 . We are now in a position to apply Lemma 7.8.

Recall that we are assuming x is an interior point of \mathcal{R}_φ . Let $\delta_0 > 0$ be such that the ρ -ball of radius δ_0 about x lies in \mathcal{R}_φ . Set $\delta = \delta_0/3$ and choose $\varepsilon > 0$ as in the lemma. Let $A, B \in K$ be such that (i) $q_k^{-2\delta} < |A| < q_k^{-\delta}$, (ii) $1 < |B| < q_k^{\min\{\varepsilon, \delta_0/6\}}$, and (iii) there exists $y \in \mathbf{P}_k^1$ such that $\zeta_{B,|A|} = \sigma_1^{-1}(\iota(y))$. This last condition is possible because $\sigma_1^{-1}(\iota(\mathcal{B}_x(\vec{v})^-))$ is a connected subset of $\mathcal{B}_{\zeta_{K,0,1}}(\vec{\infty})^-$ and shares the same boundary point. Then $y \notin \mathcal{R}_\varphi$ by the lemma. But we also find that

$$\rho(x, y) = \rho(\zeta_{K,0,1}, \zeta_{B,|A|}) = 2 \log_{q_k} |B| - \log_{q_k} |A| < 2 \log_{q_k} |B| + 2\delta < \delta_0.$$

Hence $y \in \mathcal{R}_\varphi$ by our choice of δ_0 . This contradiction completes the proof. \square

8 The Locus of Total Ramification

Definition 8.1. Let $\varphi \in k(z)$ be a nonconstant rational function. A point $x \in \mathbf{P}^1$ is said to be **totally ramified** for φ if $m_\varphi(x) = \deg(\varphi)$. The **locus of total ramification** for φ is defined as

$$\mathcal{R}_\varphi^{\text{tot}} = \{x \in \mathbf{P}^1 : m_\varphi(x) = \deg(\varphi)\}.$$

Any map of degree 2 admits a critical point, which must necessarily have multiplicity 2. Thus $\mathcal{R}_\varphi^{\text{tot}} \neq \emptyset$ when $\deg(\varphi) = 2$. But when $\deg(\varphi) \geq 3$, the locus of total ramification may be empty.

Theorem 8.2. *Let $\varphi \in k(z)$ be a nonconstant rational function. The locus of total ramification $\mathcal{R}_\varphi^{\text{tot}}$ is a closed and connected subset of the ramification locus \mathcal{R}_φ . If $\mathcal{R}_\varphi^{\text{tot}} \neq \emptyset$, then \mathcal{R}_φ is connected and contains $\text{Hull}(\text{Crit}(\varphi))$. In particular, if $\mathcal{R}_\varphi^{\text{tot}}$ is nonempty, and if $p = 0$ or $p > \deg(\varphi)$, then $\mathcal{R}_\varphi = \text{Hull}(\text{Crit}(\varphi))$.*

Proof. The result is trivial if $\mathcal{R}_\varphi^{\text{tot}} = \emptyset$ or if $\deg(\varphi) = 1$, so we will assume that we are in neither of these cases in what follows.

Suppose $\zeta \in \mathbf{P}^1$ is totally ramified for φ . Let $c \in \mathcal{R}_\varphi \setminus \{\zeta\}$, and let $x \in \mathbf{P}^1$ be any point on the open segment (ζ, c) . Then x is of type II or III. Write \mathcal{B} for the open Berkovich disk with boundary point x and containing c . Then the image $\varphi(\mathcal{B})$ does not contain $\varphi(\zeta)$, and hence cannot be equal to \mathbf{P}^1 , so the multiplicities satisfy $m_\varphi(x) \geq m_\varphi(c) > 1$ (Proposition 3.11). Thus the ramification locus is connected. Taking c to be a critical point of φ , we also see that every point in the connected hull of the critical points is a ramified point. This proves the second statement of the theorem.

Now repeat the argument in the previous paragraph with c a totally ramified point, so that $m_\varphi(x) \geq m_\varphi(c) = \deg(\varphi)$ as well. This proves connectedness of the locus of total ramification. The fact that $\mathcal{R}_\varphi^{\text{tot}}$ is closed is a consequence of semicontinuity of m_φ (Proposition 3.4(1)).

The final statement follows from Proposition 6.7 and what we have already shown. \square

Remark 8.3. In the presence of a totally ramified point, one can also use Proposition

Recall that two rational functions $\varphi, \psi \in k(z)$ are **equivalent** if there exist $\sigma_1, \sigma_2 \in \text{PGL}_2(k)$ such that $\varphi = \sigma_2 \circ \psi \circ \sigma_1$.

Corollary 8.4. *Let $\varphi \in k(z)$ be a nonconstant rational function that is equivalent to one of the following:*

1. *a polynomial or*
2. *a map with good reduction (i.e., $\deg(\varphi) = \deg(\tilde{\varphi})$).*

Then the ramification locus of φ is connected and contains $\text{Hull}(\text{Crit}(\varphi))$. If $p = 0$ or $p > \deg(\varphi)$, then $\mathcal{R}_\varphi = \text{Hull}(\text{Crit}(\varphi))$.

Proof. If φ is a polynomial, then $\infty \in \mathbb{P}^1(k)$ is totally ramified, and the theorem applies. If φ has good reduction, then the Gauss point is totally ramified for φ , and we may again use the theorem. The conclusions of the corollary are invariant under change of equivalence class representative (Corollary 3.7), so the proof is complete. \square

It is quite special for there to be a totally ramified point lying off the connected hull of the critical points.

Proposition 8.5. *Let $\varphi \in k(z)$ be a nonconstant rational function such that $\mathcal{R}_\varphi^{\text{tot}} \cap [\mathbf{P}^1 \setminus \text{Hull}(\text{Crit}(\varphi))]$ is nonempty. Then $0 < p \leq \deg(\varphi)$ and $\deg(\varphi) \equiv 0$ or $1 \pmod{p}$. Moreover, if $p \nmid \deg(\varphi)$, then $\mathcal{R}_\varphi^{\text{tot}}$ is reduced to a point.*

Remark 8.6. The rational function $\varphi(z) = (z^{p+1} + pz + 1)/z$ illustrates the behavior described by the proposition. Indeed, the Algebraic Reduction Formula shows $\mathcal{R}_\varphi^{\text{tot}} = \{\zeta_{0,1}\}$, while the critical points all lie outside the disk $D(0, |p|^{-1/(p+1)})^-$.

Proof. As $\mathcal{R}_\varphi \not\subset \text{Hull}(\text{Crit}(\varphi))$, we see $0 < p \leq \deg(\varphi)$ by Proposition 6.7.

Without loss of generality, we may replace k by an algebraically closed and complete extension in order to assume that \mathbf{P}_k^1 has no point of type III or IV. Let $x \in \mathbf{P}^1 \setminus \text{Hull}(\text{Crit}(\varphi))$ be a totally ramified point, which must be of type II. Then x is either an interior point or an end point of the ramification locus (Corollary 7.6). In the former case, we know φ has inseparable reduction at x (Proposition 7.9), so that $p \mid m_\varphi(x) = \deg(\varphi)$.

Suppose now that $x \in \mathcal{R}_\varphi^{\text{tot}} \setminus \text{Hull}(\text{Crit}(\varphi))$ is a type II endpoint of the ramification locus. After a change of coordinates on the source and target, we may assume that $x = \zeta_{0,1} = \varphi(\zeta_{0,1})$. Then φ has good reduction, and its reduction $\tilde{\varphi}$ has a unique critical point corresponding the unique ramified direction at the Gauss point (Algebraic Reduction Formula). We now appeal to [5, Cor. 1.3] to conclude that

$$\deg(\varphi) = \deg(\tilde{\varphi}) \equiv 0 \text{ or } 1 \pmod{p}.$$

Finally, let V be a connected component of $\mathbf{P}^1 \setminus \text{Hull}(\text{Crit}(\varphi))$ containing a totally ramified point y_1 . Suppose there exists a second point $y_2 \neq y_1$ in $\mathcal{R}_\varphi^{\text{tot}}$. As $\mathcal{R}_\varphi^{\text{tot}}$ is connected, we may further assume $y_2 \in V$. Then the whole segment $[y_1, y_2]$ is contained in $\mathcal{R}_\varphi^{\text{tot}} \cap V$ by connectedness. Select a type II point $y \in (y_1, y_2)$. Proposition 6.6 shows that φ must have inseparable reduction at y . Hence $p \mid m_\varphi(y) = \deg(\varphi)$. \square

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